Institute of Computer Science
Academy of Sciences of the Czech Republic

Calculus Digest

Jiří Rohn
http://uivtx.cs.cas.cz/~rohn

Technical report No. V-1154

02.02.2012

Pod Vodárenskou věží 2, 182 07 Prague 8, phone: +420 266 051 111, fax: +420 286 585 789,
e-mail:rohn@cs.cas.cz

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Abstract:

This text was originally prepared for students of the "Business Mathematics II" class at the Anglo-American University in Prague. It is written in "one-topic-one-page" style, where each topic is allotted the space of one page only.

Keywords:
Function of one variable, limit, continuity, derivative, minima and maxima, plotting, definite integral, indefinite integral, integration by parts and by substitution, function of two variables.

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Basic facts (primary school level)

1. $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$
2. $\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$
3. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
4. $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$
5. $\frac{a}{0}$ is not defined (no result)
6. $(a+b)(c+d) = ac + ad + bc + bd$
7. $(a+b)^2 = a^2 + 2ab + b^2$
8. $(a-b)^2 = a^2 - 2ab + b^2$
9. $a^2 - b^2 = (a+b)(a-b)$
10. $ax^2 + bx + c = 0$ has roots $x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, and there holds $ax^2 + bx + c = a(x-x_1)(x-x_2)$

Two often misunderstood facts.

1. $\sqrt{x}$ is a *nonnegative*[2] number y satisfying $x = y^2$
2. $|a| = a$ if $a \geq 0$, and $|a| = -a$ if $a < 0$

[2]Thus, $\sqrt{4} = 2$, not -2, despite the fact that $(-2)^2 = 4$.

Intervals

1. (a, b) is the set of all numbers satisfying $a < x < b$ (open interval)
2. $[a, b]$ is the set of all numbers satisfying $a \leq x \leq b$ (closed interval)
3. $(a, b]$ is the set of all numbers satisfying $a < x \leq b$ (half-closed interval)
4. $[a, b)$ is the set of all numbers satisfying $a \leq x < b$ (half-closed interval)

Elementary functions and their domains

1. $y = e^x$ $\quad (D = (-\infty, \infty))$
2. $y = \ln x$ $\quad (D = (0, \infty))$
3. $y = x^a$ $\quad$ (domain depends on a)
4. $y = \sin x$ $\quad (D = (-\infty, \infty))$
5. $y = \cos x$ $\quad (D = (-\infty, \infty))$
6. $y = \tan x$ $\quad (D = (-\infty, \infty)$ except all $x = \frac{k\pi}{2}$, k odd integer)
7. $y = \cot x$ $\quad (D = (-\infty, \infty)$ except all $x = \frac{k\pi}{2}$, k even integer)

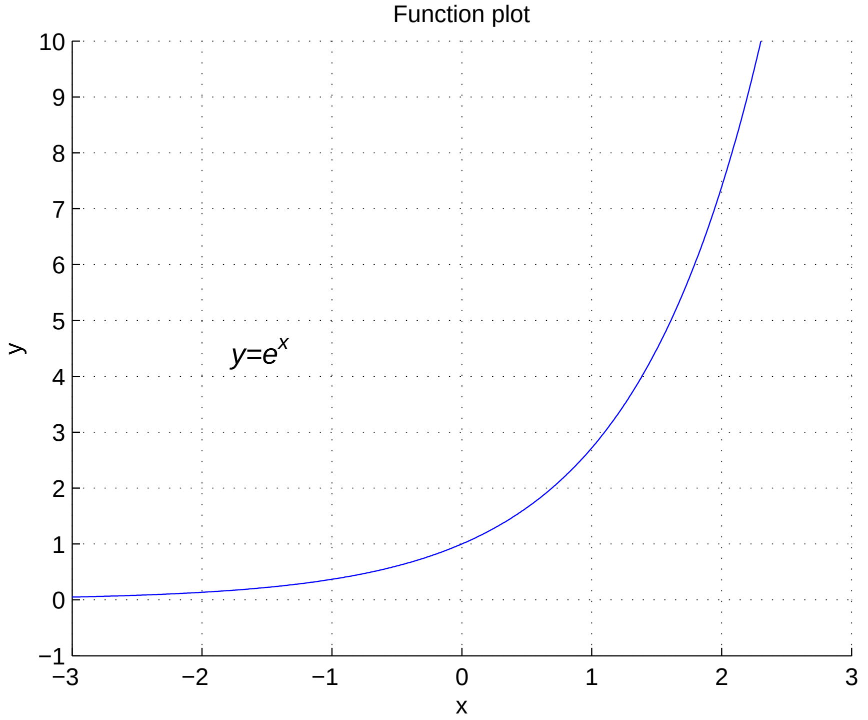
Functions a^x and $\log_a x$ can be expressed via e^x and $\ln x$ (see p. 4, items 8 and 9).

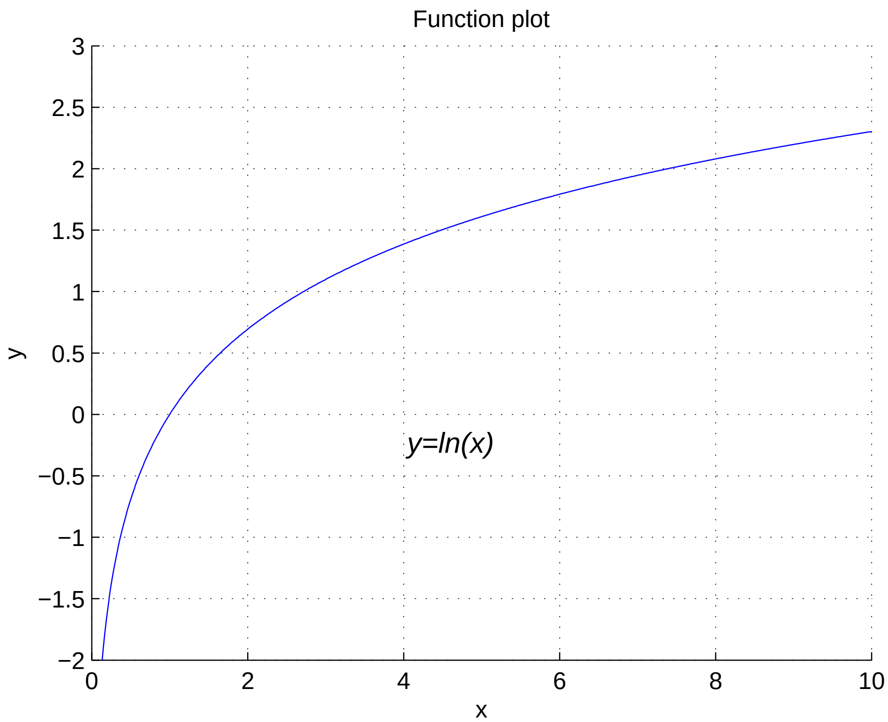
1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \dots$
2. $\ln x = 2\left(\frac{x-1}{x+1} + \frac{1}{3}(\frac{x-1}{x+1})^3 + \frac{1}{5}(\frac{x-1}{x+1})^5 + \frac{1}{7}(\frac{x-1}{x+1})^7 + \dots\right)$
3. $x^a = e^{a \cdot \ln x}$
4. $\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots$
5. $\cos x = \sin(\frac{\pi}{2} - x)$
6. $\tan x = \frac{\sin x}{\cos x}$
7. $\cot x = \frac{\cos x}{\sin x}$
8. $a^x = e^{(\ln a)x}$
9. $\log_a x = \frac{\ln x}{\ln a}$

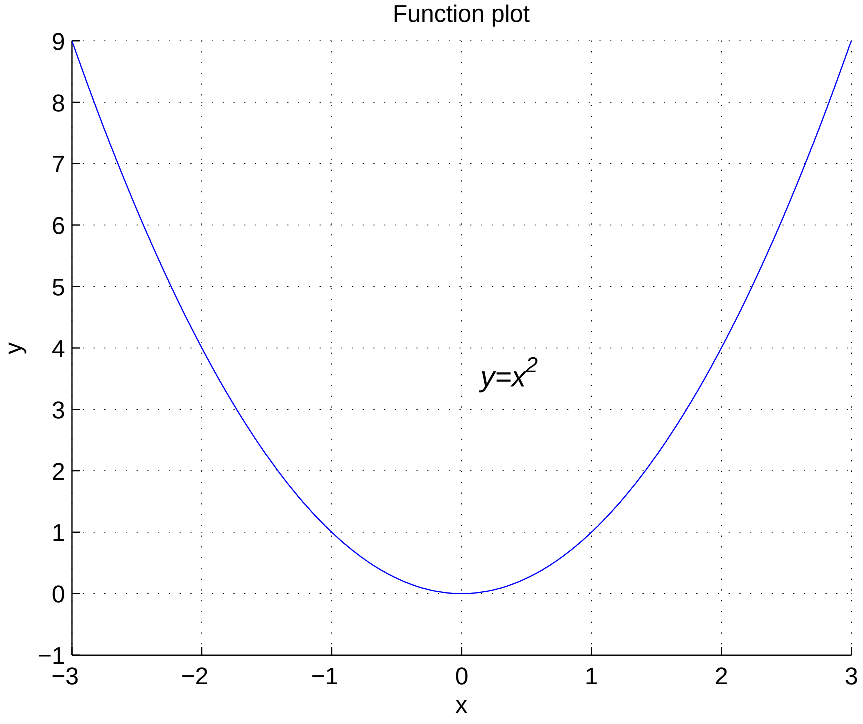
Two important numbers:

1. $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \dots = 2.71828\dots$
2. $\pi = 4 \cdot (\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots) = 3.14159\dots$

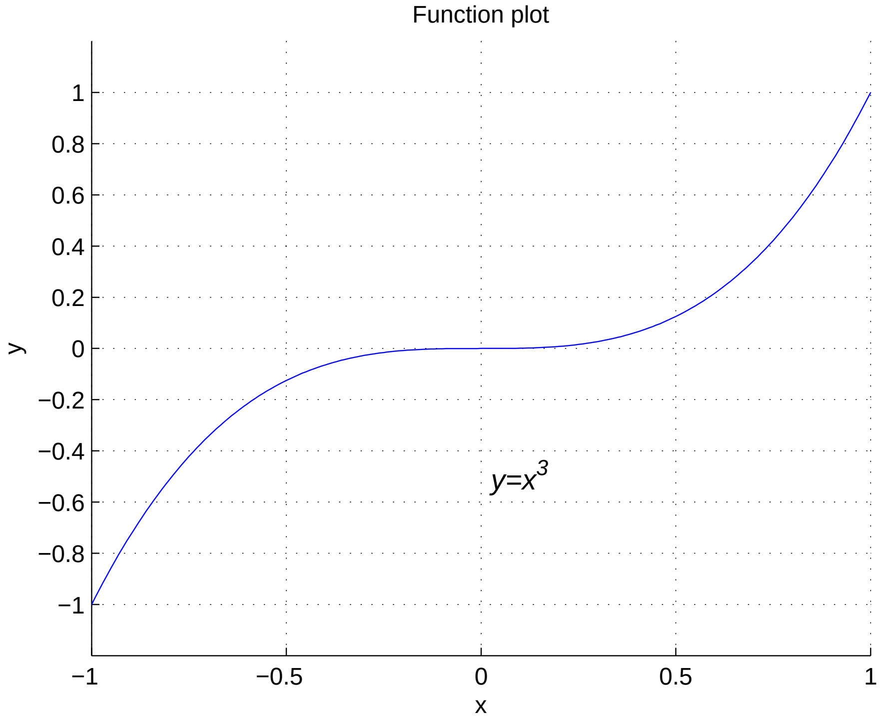
$(n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n)$


Function plot
y=e^x
y
x
10
9
8
7
6
5
4
3
2
1
0
−1
−3
−2
−1
0
1
2
3

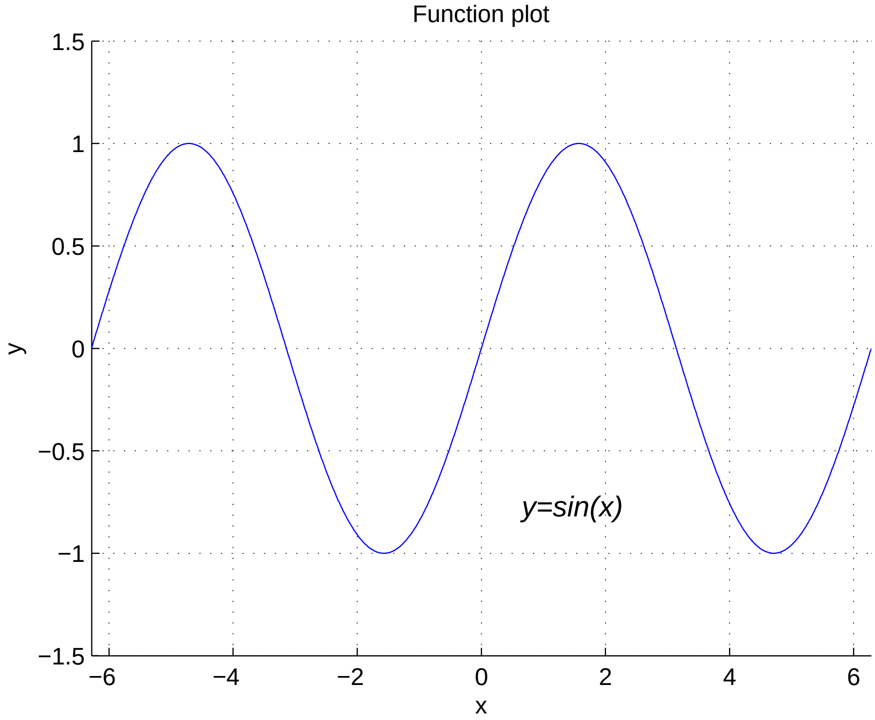

Function plot
y=ln(x)
y
x
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
0
2
4
6
8
10

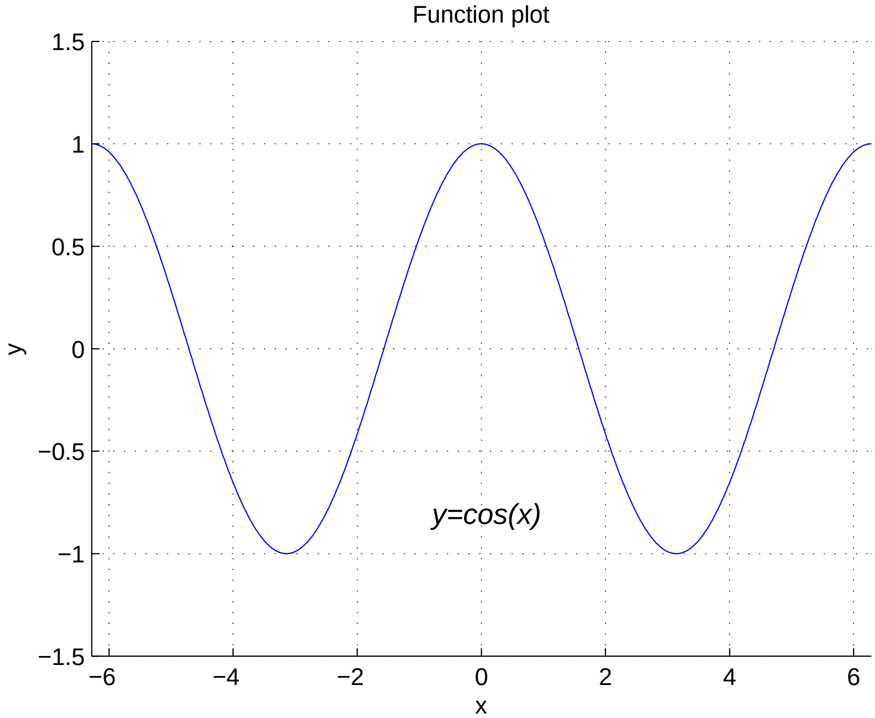


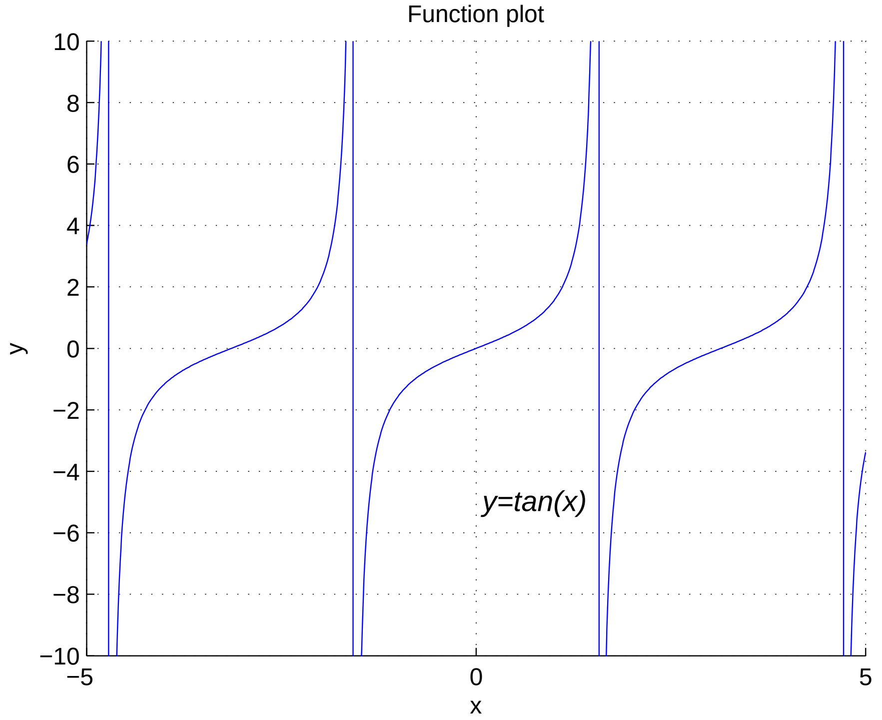
Function plot
9
8
7
6
5
4
3
2
1
0
−1
y
$y=x^2$
−3
−2
−1
0
1
2
3
x

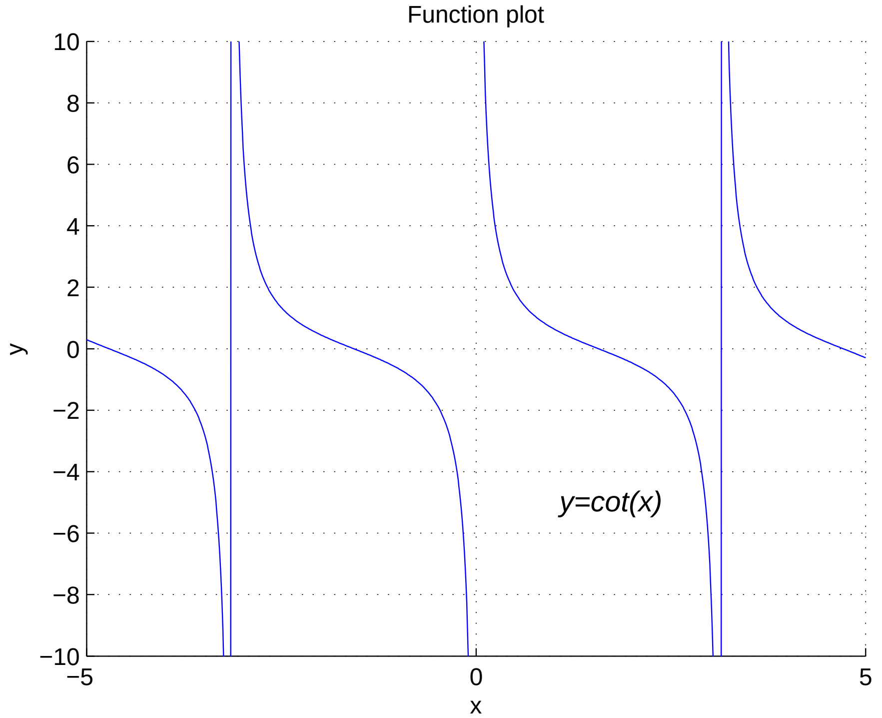


Function plot
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
y
$y=x^3$
−1
−0.5
0
0.5
1
x


Function plot
1.5
1
0.5
0
−0.5
−1
−1.5
y
−6
−4
−2
0
2
4
6
x
y=sin(x)


Function plot
1.5
1
0.5
0
−0.5
−1
−1.5
y
−6
−4
−2
0
2
4
6
x
y=cos(x)


Function plot
10
8
6
4
2
0
−2
−4
−6
−8
−10
y
−5
0
5
x
y=tan(x)


Function plot
10
8
6
4
2
0
−2
−4
−6
−8
−10
y
−5
0
5
x
y=cot(x)

Properties of elementary functions (secondary school level)

1. $e^x > 0$

2. $e^x e^y = e^{x+y}$

3. $\frac{e^x}{e^y} = e^{x-y}$

4. $(e^x)^y = e^{x \cdot y}$

5. for each $x > 0$ there exists exactly one y satisfying $x = e^y$ (namely, $y = \ln x$)

6. $x = e^{\ln x}$

7. $\ln(x \cdot y) = \ln x + \ln y$

8. $\ln \frac{x}{y} = \ln x - \ln y$

9. $\ln x^y = y \ln x$

10. $x^0 = 1$

11. $x^{-a} = \frac{1}{x^a}$, in particular $x^{-1} = \frac{1}{x}$

12. $x^{\frac{m}{n}} = \sqrt[n]{x^m}$, in particular $x^{\frac{1}{2}} = \sqrt{x}$

13. $(xy)^a = x^a y^a$

14. $\sin(x + 2\pi) = \sin x$

15. $\cos(x + 2\pi) = \cos x$

16. $\sin^2 x + \cos^2 x = 1$

17. $\tan(x + \pi) = \tan x$

18. $\cot(x + \pi) = \cot x$

19. $\cot x = \frac{1}{\tan x}$

Functions

The functions we meet in examples are constructed from elementary functions by repeated use of five operations:

$$f(x) + g(x)$$

$$f(x) - g(x)$$

$$f(x) \cdot g(x)$$

$$\frac{f(x)}{g(x)}$$

$$f(g(x))$$

the last of them being called the *composite* function (as e.g. $\sin(x^2)$).

Definition.
A function $f(x)$ is said to have limit d at a point c, which we write as

$$\lim_{x \to c} f(x) = d, \tag{0.1}$$

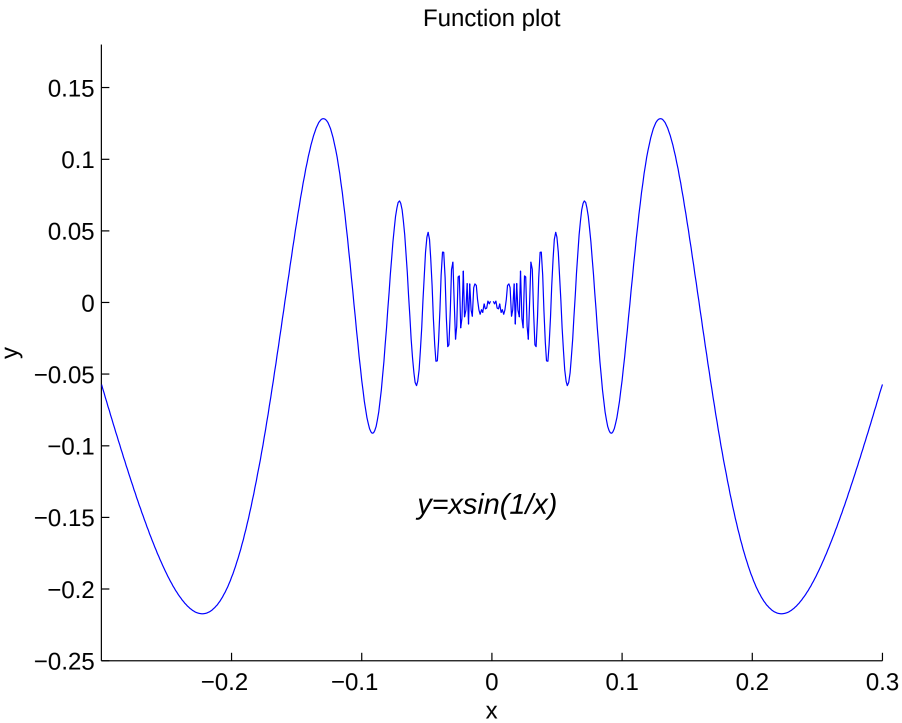
if $f(x)$ approaches d as x approaches c (without touching c).

Explanation. This is an informal definition since the word "approaches" can be understood only intuitively. The exact definition[3] is beyond the scope of a business mathematics class. The words "without touching c" mean that the possibility of $x = c$ is excluded. As a consequence, $f(x)$ need not be defined at c, yet $\lim_{x \to c} f(x)$ may exist.

Example. The function

$$f(x) = x \sin \tfrac{1}{x}$$

is obviously not defined at 0, yet $\lim_{x \to 0} f(x) = 0$, as shown by its graph:



[3]Exact definition of (0.1): for each $\varepsilon > 0$ there exists a $\delta > 0$ such that each x with $0 < |x - c| < \delta$ satisfies $|f(x) - d| < \varepsilon$.

Definition.
A function $f(x)$ is called *continuous* in an interval I (open, closed, or half-closed) if

$$\lim_{x \to c} f(x) = f(c)$$

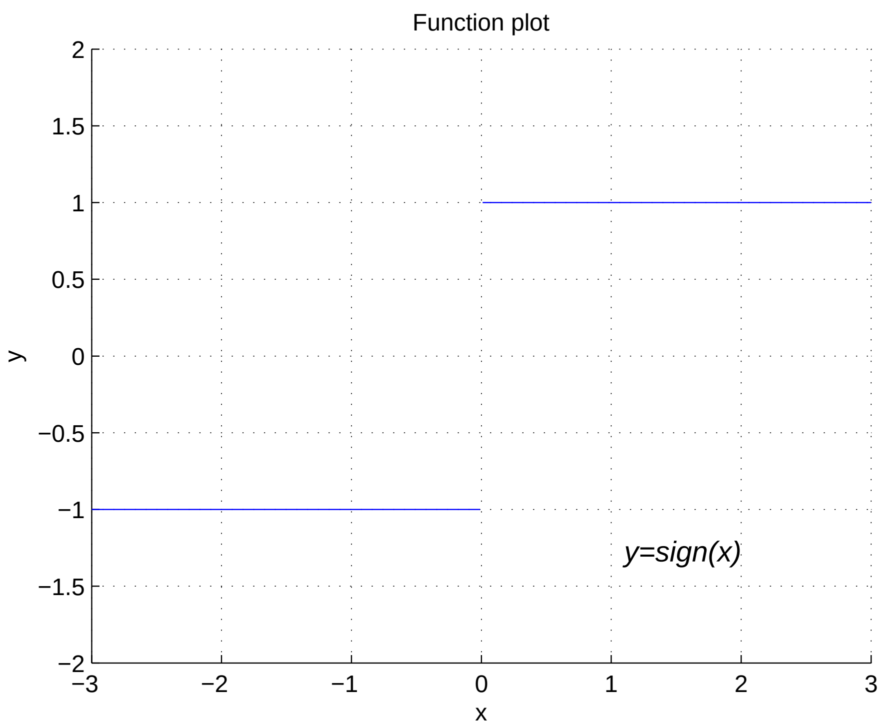
for each $c \in I$.[4]

Important fact.
All elementary functions are continuous in their domains.

Explanation. A continuous function "does not jump" at any point; its graph can be drawn up without lifting the pen from the paper. An example of a discontinuous function is the sign function defined by

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0 \end{cases}$$

which "jumps" at $c = 0$. If x approaches 0 from the left, then $\operatorname{sign}(x)$ approaches -1, while if x approaches 0 from the right, then $\operatorname{sign}(x)$ approaches 1:



[4] Which means that for each $c \in I$, $f(x)$ approaches $f(c)$ as x approaches c.

The derivative of a function f at x is formally **defined** as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

This means that the value of

$$\frac{f(x+h) - f(x)}{h}$$

approaches $f'(x)$ as h approaches 0.

Note. Instead of $f'(x)$, we can also alternatively write $\frac{df}{dx}$. The meaning is the same.

Example 1. For the quadratic function $f(x) = x^2$ we have

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = 2x + h$$

and this value approaches $2x$ as h approaches 0. Hence,

$$(x^2)' = 2x.$$

Example 2. For the reciprocal function $f(x) = \frac{1}{x}$ we have

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{-1}{(x+h)x}$$

and this value approaches $-\frac{1}{x^2}$ as h approaches 0. Hence,

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}.$$

Derivatives of elementary functions

1. $(e^x)' = e^x$
2. $(\ln x)' = \frac{1}{x}$
3. $(x^a)' = ax^{a-1}$
   (particular cases: $1' = 0$, $x' = 1$; includes roots: $(\sqrt{x})' = (x^{\frac{1}{2}})'$, etc.)
4. $(\sin x)' = \cos x$
5. $(\cos x)' = -\sin x$
6. $(\tan x)' = \frac{1}{\cos^2 x}$
7. $(\cot x)' = -\frac{1}{\sin^2 x}$

Additionally, we have

8. $(a^x)' = a^x \cdot \ln a$
9. $(\log_a x)' = \frac{1}{x \cdot \ln a}$
10. $c' = 0$

These formulae should be **memorized**; this is the "alphabet" of calculus.

$$
\begin{aligned}
(f(x)+c)' &= f'(x) \qquad \text{(additive constant)} && (0.2)\\
(c\cdot f(x))' &= c\cdot f'(x) \quad \text{(multiplicative constant)} && (0.3)\\
(f(x)+g(x))' &= f'(x)+g'(x) && (0.4)\\
(f(x)-g(x))' &= f'(x)-g'(x) && (0.5)\\
(f(x)\cdot g(x))' &= f'(x)\cdot g(x)+f(x)\cdot g'(x) && (0.6)\\
\left(\frac{f(x)}{g(x)}\right)' &= \frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{g^2(x)} && (0.7)\\
(f(g(x)))' &= f'(g(x))\cdot g'(x) && (0.8)
\end{aligned}
$$

Explanation to rule (0.8). To compute the derivative of the composite function

$$h(x) = f(g(x))$$

write it in the form

$$h(x) = f(y), \quad y = g(x)$$

and use the formula

$$h'(x) = f'(y)\cdot y' \tag{0.9}$$

i.e., differentiate the function $f(y)$ with respect to y and multiply the result by the derivative of $g(x)$

$$h'(x) = f'(y)\cdot g'(x)$$

then substitute back $y = g(x)$:

$$h'(x) = f'(g(x))\cdot g'(x).$$

In this way we get the right-hand side of (0.8).

Example. To compute the derivative of the composite function

$$h(x) = \ln(1+x^2),$$

we write it in the form

$$h(x) = \ln(y), \quad y = 1+x^2$$

and use the formula (0.9)

$$h'(x) = (\ln(y))'\cdot y' = \frac{1}{y}\cdot 2x,$$

then we substitute back $y = 1+x^2$:

$$h'(x) = \frac{2x}{1+x^2}.$$

Examples of using the rules (0.2)-(0.8):

1. $f(x) = \sin x + 5$
   $f'(x) = (\sin x)' = \cos x$
   (rule (0.2))

2. $f(x) = 3 \cdot \ln x$
   $f'(x) = 3 \cdot (\ln x)' = 3 \cdot \frac{1}{x} = \frac{3}{x}$
   (rule (0.3))

3. $f(x) = x^2 + 2x + 1$
   $f'(x) = (x^2)' + (2x)' + 1' = 2x + 2 \cdot 1 + 0 = 2x + 2$
   (rule (0.4))

4. $f(x) = e^x - \cos x$
   $f'(x) = (e^x)' - (\cos x)' = e^x - (-\sin x) = e^x + \sin x$
   (rule ())

5. $f(x) = x \cdot \ln x$
   $f'(x) = x' \cdot \ln x + x \cdot (\ln x)' = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$
   (rule (0.6))

6. $f(x) = \frac{x^2-1}{x^2+1}$
   $f'(x) = \frac{(x^2-1)'(x^2+1)-(x^2-1)(x^2+1)'}{(x^2+1)^2} = \frac{2x(x^2+1)-(x^2-1)2x}{(x^2+1)^2} = \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$
   (rule (0.11))

7. $f(x) = \sin(x^4)$: write $f(x) = \sin(y)$, $y = x^4$
   $f'(x) = (\sin(y))' \cdot y' = \cos(y) \cdot (x^4)' = \cos(x^4) \cdot 4x^3 = 4x^3 \cos(x^4)$
   (rule (0.8))

Definitions.
A function f is said to have a *relative minimum* at c if it satisfies

$$f(x) \geq f(c)$$

for each x in some neighborhood of c (an open interval containing c).

A function f is said to have a *relative maximum* at c if it satisfies

$$f(x) \leq f(c)$$

for each x in some neighborhood of c (an open interval containing c).

Facts.

1. If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has a *relative minimum* at c.
2. If $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has a *relative maximum* at c.
3. If $f'(c) \neq 0$, then $f(x)$ has neither relative minimum, nor relative maximum at c.

A point c at which $f'(c) = 0$ is called a *critical point*.

Definitions.
A function f is said to be *increasing* in an open interval (a, b) if

$$f(x_1) < f(x_2)$$

for each $a < x_1 < x_2 < b$ (as e.g. x^2 in $(0, \infty)$, see p. 6).

A function f is said to be *decreasing* in an open interval (a, b) if

$$f(x_1) > f(x_2)$$

for each $a < x_1 < x_2 < b$ (as e.g. x^2 in $(-\infty, 0)$, see p. 6).

A function f is said to be *convex* in an open interval (a, b) if it is "hollowed down" there (as e.g. e^x in $(-\infty, \infty)$, see p. 5).

A function f is said to be *concave* in an open interval (a, b) if it is "hollowed up" there (as e.g. $\ln x$ in $(0, \infty)$, see p. 5).

Facts.

1. If $f'(x) > 0$ for each $x \in (a, b)$, then $f(x)$ is *increasing* in (a, b).
2. If $f'(x) < 0$ for each $x \in (a, b)$, then $f(x)$ is *decreasing* in (a, b).
3. If $f''(x) > 0$ for each $x \in (a, b)$, then $f(x)$ is *convex* in (a, b).
4. If $f''(x) < 0$ for each $x \in (a, b)$, then $f(x)$ is *concave* in (a, b).

Summary:

	$f'' > 0$	$f'' < 0$
$f' = 0$	minimum	maximum
$f' > 0$	increasing, convex	increasing, concave
$f' < 0$	decreasing, convex	decreasing, concave

Example. The function

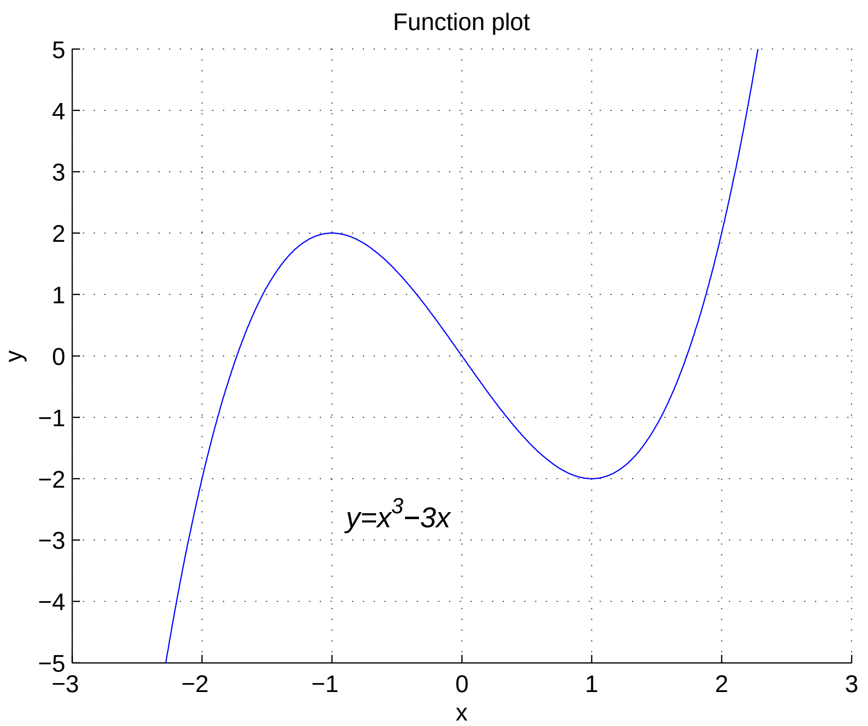
$$f(x) = x^3 - 3x$$

has

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1),$$

$$f''(x) = 3 \cdot 2x = 6x,$$

hence $f'(c) = 0$ for $c_1 = -1$ and $c_2 = 1$.
Since $f''(c_1) = -6 < 0$ and $f''(c_2) = 6 > 0$, f has a relative maximum at c_1 and a relative minimum at c_2.
Since $f'(x) > 0$ for $x^2 > 1$ and $f'(x) < 0$ for $x^2 < 1$, f is increasing in $(-\infty, -1)$ and in $(1, \infty)$, and decreasing in $(-1, 1)$.
Since $f''(x) > 0$ for $x > 0$ and $f''(x) < 0$ for $x < 0$, f is convex in $(0, \infty)$, and concave in $(-\infty, 0)$.



Given a function $f(x)$ on a *closed* interval $[a, b]$ and an integer $n \geq 1$, define the nth integral sum S_n as follows: first compute

$$h = \frac{b-a}{n}$$

(the so-called step), and then evaluate

$$S_n = f(a+h)h + f(a+2h)h + f(a+3h)h + \ldots + f(a+nh)h \tag{0.10}$$

Definition.
If $f(x)$ is continuous on $[a, b]$ (see p. 12), then, as n approaches infinity (i.e., increases without bound), S_n is guaranteed to approach certain number which is denoted by

$$\int_a^b f(x)dx$$

and is called the *definite* integral of $f(x)$ over the interval $[a, b]$.

Note 1. dx is only a symbol which has evolved from Δx, the 17th century notation for the above h; in this way the integral symbol "copies" the form of the integral sum (0.10).

Note 2. The above definition holds for *continuous* functions only. For the general case a more complicated way is needed.

Note 3. If $f(x)$ is nonnegative in $[a, b]$, then $\int_a^b f(x)\,dx$ expresses the *area* of the region bounded by the curve $y = f(x)$ and by the lines $x = a$, $x = b$, $y = 0$. For example, $\int_{-1}^{1} \sqrt{1-x^2}\,dx$ is equal to the area of the half-circle centered in the origin of the plane and having radius 1 and $y \geq 0$, so that its value is $\frac{\pi}{2}$.

The fundamental theorem of calculus, and definition of indefinite integral

The fundamental theorem of calculus, coauthored by I. Newton and G. W. Leibniz (second half of the 17th century), asserts that

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is any function with the property

$$F'(x) = f(x) \quad \text{for each } x \in [a,b]. \tag{0.11}$$

Definition.
A function $F(x)$ with the above property is called an *indefinite* integral[5] of $f(x)$ over $[a,b]$ and is denoted by

$$\int f(x)\,dx$$

(i.e., the same symbol as for the definite integral, but without bounds).

Note 1. Thus, the definite integral is a *number* whereas indefinite integral is a *function*; the definite integral can be computed as soon as we know a corresponding indefinite integral. Next pages of this text are dedicated to computation of indefinite integrals.

Note 2. The definite integral, if it exists, is uniquely determined. On the contrary, an indefinite integral, if it exists, is *never* unique: in fact, if $F(x)$ satisfies (0.11), then so does $F(x) + C$ for any constant C because

$$(F(x) + C)' = F'(x) + C' = F(x) + 0 = F(x) = f(x).$$

Therefore, we always add "$+C$" to the computed indefinite integral $F(x)$ to indicate that each function formed by adding a constant C to $F(x)$ is also an indefinite integral, as e.g. in

$$\int \cos x\,dx = \sin x + C$$

etc.

Note 3. Another fundamental theorem (although usually not quoted as such) says that *if $f(x)$ is continuous in an open interval (c,d), then it has an indefinite integral $F(x)$ in (c,d).* This theorem states only *existence* of an indefinite integral; it does not show a way how to find it.

[5] Or, *primitive function.*

Integrals of elementary functions

1. $\int e^x \, dx = e^x + C$
2. $\int \ln x \, dx = x(\ln x - 1) + C$
3. $\int x^a \, dx = \frac{x^{a+1}}{a+1} + C$ if $a \neq -1$,
   $\int \frac{1}{x} \, dx = \ln |x| + C$ if $a = -1$ (notice the absolute value[6])
4. $\int \sin x \, dx = -\cos x + C$
5. $\int \cos x \, dx = \sin x + C$
6. $\int \tan x \, dx = -\ln |\cos x| + C$
7. $\int \cot x \, dx = \ln |\sin x| + C$

Related results:

1. $\int 0 \, dx = C$
2. $\int 1 \, dx = x + C$
3. $\int c \, dx = cx + C$
4. $\int a^x \, dx = \frac{a^x}{\ln a} + C$
5. $\int \log_a x \, dx = \frac{x(\ln x - 1)}{\ln a} + C$
6. $\int \frac{1}{\cos^2 x} \, dx = \tan x + C$
7. $\int \frac{1}{\sin^2 x} \, dx = -\cot x + C$
8. $\int \frac{1}{1+x^2} \, dx = \arctan x + C$
   (a new function $y = \arctan x$ defined in $D = (-\infty, \infty)$ by $x = \tan y$, $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$)
9. $\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$
   (a new function $y = \arcsin x$ defined in $D = [-1, 1]$ by $x = \sin y$, $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$)

[6] Which is often wrongly omitted.

Integration rules: addition and subtraction

$$\begin{aligned}\int c\cdot f(x)\,dx &= c\cdot\int f(x)\,dx \quad \text{(multiplicative constant)}\\ \int (f(x)+g(x))\,dx &= \int f(x)\,dx + \int g(x)\,dx\\ \int (f(x)-g(x))\,dx &= \int f(x)\,dx - \int g(x)\,dx\end{aligned}$$

Examples:

1. $\int (3x^5 - \sin x + \sqrt{x})\,dx = 3\int x^5\,dx - \int \sin x\,dx + \int x^{\frac{1}{2}}\,dx = 3\frac{x^6}{6} - (-\cos x) + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{1}{2}x^6 + \cos x + \frac{2}{3}x^{\frac{3}{2}} + C$
2. $\int \frac{x^2+2x+3}{x^2}\,dx = \int (1+\frac{2}{x}+3x^{-2})\,dx = \int 1\,dx + 2\int \frac{1}{x}\,dx + 3\int x^{-2}\,dx = x + 2\ln|x| + 3\frac{x^{-1}}{-1} = x + 2\ln|x| - \frac{3}{x} + C$
3. $\int \tan^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx = \int (\frac{1}{\cos^2 x} - 1)\,dx = \int \frac{1}{\cos^2 x}\,dx - \int 1\,dx = \tan x - x + C$
4. $\int_0^1 x^2\,dx$. Indefinite integral: $F(x) = \int x^2\,dx = \frac{x^3}{3}$. By Newton-Leibniz formula, $\int_0^1 x^2\,dx = F(1) - F(0) = \frac{1}{3}$

$$\int f'(x) \cdot g(x)\, dx = f(x) \cdot g(x) - \int f(x) \cdot g'(x)\, dx$$

Examples:

1. $\int e^x x\, dx = \int (e^x)' x\, dx = e^x x - \int e^x \cdot x'\, dx = e^x x - \int e^x\, dx = e^x x - e^x = e^x(x-1) + C$
2. $\int x \cos x\, dx = \int (\cos x) x\, dx = \int (\sin x)' x\, dx = (\sin x) x - \int (\sin x) \cdot 1\, dx = x \sin x - (-\cos x) = x \sin x + \cos x + C$
3. $\int \ln x\, dx = \int 1 \cdot \ln x\, dx = \int x' \ln x\, dx = x \ln x - \int x (\ln x)'\, dx = x \ln x - \int x \frac{1}{x}\, dx = x \ln x - \int 1\, dx = x \ln x - x = x(\ln x - 1) + C$. (See p. 22, item 2.)

Explanation. The original integral is given in the form

$$\int h(x) \cdot k(x)\, dx$$

It is not said which one of the two functions should be taken for $f'(x)$ and $g(x)$, respectively: we must make the choice. In the process, $f'(x)$ integrated and $g(x)$ is differentiated. Therefore, we must choose $f'(x)$ to be a function among $h(x)$, $k(x)$ which we are able to integrate, and if we are able to do so with both of them, to choose $g(x)$ as a function which simplifies by differentiation.

$$\int f(x)\,dx = \int f(\varphi(t)) \cdot \varphi'(t)\,dt$$

Explanation. We make substitution $x = \varphi(t)$, where φ must be increasing or decreasing. From $x = \varphi(t)$ we have

$$\frac{dx}{dt} = \varphi'(t)$$

so that we replace dx in the original integral by $\varphi'(t)dt$. Then we compute the integral

$$\int f(\varphi(t)) \cdot \varphi'(t)\,dt$$

and finally we must replace the auxiliary variable t by the original variable x. This is done by solving first the equation $x = \varphi(t)$ for t, thus obtaining $t = \psi(x)$, where ψ is a certain function. Then we replace t by $\psi(x)$ everywhere in the result (see Examples 1-3 below). Sometimes it helps to evaluate $\frac{dt}{dx}$ instead of $\frac{dx}{dt}$ (Examples 4 and 5).

Examples:

1. $\int \cos 4x\,dx$. We choose substitution $4x = t$, $x = \frac{1}{4}t$, $\frac{dx}{dt} = \frac{1}{4}$, $dx = \frac{1}{4}dt$.
   Then $\int \cos 4x\,dx = \int \cos t \cdot \frac{1}{4}\,dt = \frac{1}{4}\int \cos t\,dt = \frac{1}{4}\sin t = \frac{1}{4}\sin 4x + C$

2. $\int \frac{1}{3x+5}\,dx$. Substitution $3x + 5 = t$, $x = \frac{t-5}{3}$, $\frac{dx}{dt} = \frac{1}{3}$.
   Then $\int \frac{1}{3x+5}\,dx = \int \frac{1}{t} \cdot \frac{1}{3}\,dt = \frac{1}{3}\int \frac{1}{t}\,dx = \frac{1}{3}\ln|t| = \frac{1}{3}\ln|3x+5| + C$

3. $\int (x+1)^{100}\,dx$. Substitution $x + 1 = t$, $dx = dt$.
   Then $\int (x+1)^{100}\,dx = \int t^{100}\,dt = \frac{t^{101}}{101} = \frac{(x+1)^{101}}{101} + C$

4. $\int_1^2 \frac{x}{1+x^2}\,dx$. Substitution $1 + x^2 = t$, $\frac{dt}{dx} = 2x$ (notice the difference: $\frac{dt}{dx}$, not $\frac{dx}{dt}$), $x\,dx = \frac{1}{2}dt$. Then $F(x) = \int \frac{x}{1+x^2}\,dx = \int \frac{1}{t} \cdot \frac{1}{2}\,dt = \frac{1}{2}\ln|t| = \frac{1}{2}\ln|1+x^2| = \frac{1}{2}\ln(1+x^2) + C$ (because $1 + x^2$ is always positive), and $\int_1^2 \frac{x}{1+x^2}\,dx = F(2) - F(1) = \frac{1}{2}\ln 5 - \frac{1}{2}\ln 2 = \frac{1}{2}\ln\frac{5}{2}$.

5. $\int \sin^3 x \cos x\,dx$. Substitution $\sin x = t$, $\frac{dt}{dx} = \cos x$, $\cos x\,dx = dt$.
   Then $\int \sin^3 x \cos x\,dx = \int t^3\,dt = \frac{t^4}{4} = \frac{\sin^4 x}{4} + C$

Definitions.
A function $f(x, y)$ is said to have a *relative minimum* at a point (c, d) if it satisfies

$$f(x, y) \geq f(c, d)$$

for each (x, y) in some neighborhood of (c, d).

A function $f(x, y)$ is said to have a *relative maximum* at a point (c, d) if it satisfies

$$f(x, y) \leq f(c, d)$$

for each (x, y) in some neighborhood of (c, d).

Facts.

(a) If

$$f_x(c, d) = 0$$
$$f_y(c, d) = 0$$
$$f_{xx}(c, d) f_{yy}(c, d) - f_{xy}^2(c, d) > 0$$
$$f_{xx}(c, d) > 0$$

then $f(x, y)$ has a *relative minimum* at (c, d).

(b) If

$$f_x(c, d) = 0$$
$$f_y(c, d) = 0$$
$$f_{xx}(c, d) f_{yy}(c, d) - f_{xy}^2(c, d) > 0$$
$$f_{xx}(c, d) < 0$$

then $f(x, y)$ has a *relative maximum* at (c, d).

Observe that in both cases the first three conditions are the same.