The sole purpose of this paper consists in presenting the proofs to eleven theorems given in the author's paper "Solving interval linear systems" [0]. The reader is assumed to be familiar with that paper; notations, formulæ and references introduced there are used here without further explanations.

1. Theorems 0

Theorem 0. Let for each $y \in Y$ the equation
\[
\lambda^1 y + \lambda^2 y = b_y
\]
have a nonnegative solution $\lambda^1_y$, $\lambda^2_y$. Then for each $\lambda \in \lambda^1$ and $b \in b^1$, the equation $\lambda x = b$ has a solution belonging to $\text{Conv} \{\lambda^1_y, \lambda^2_y; y \in Y\}$.

Comment. As it will be seen from the proof, the theorem is valid for arbitrary non interval matrices (if $A_{y^1}$ is defined by $A_{y^1} = A_y^1 - T_y^1 A^2$, $y \in Y^1$, $b \in b^1$). The proof is constructive: an algorithm for computing a solution to $\lambda x = b$ directly from the vectors $\lambda^1_y - \lambda^2_y$ ($y \in Y$) is given below. For its description, we give two definitions.

First we define by induction an ordering for each set $Y_j = \{y \in \mathbb{R}^n; |y_k| = 1 (k = 1, \ldots, j)\}$ ($j \in \mathbb{N}$); (i) the ordering of $Y_1$ is 1, -1; (ii) if $y_1, \ldots, y_j$ is the ordering of $Y_j$, then $(y_1, 1), \ldots, (y_j, 1)$.

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\((y_1, -1), \ldots, (y_j, -1)\) is the ordering of \(Y_{j+1}\). Second, given a sequence \(x_1, \ldots, x_{2m}\), then each pair \(a_k, a_{k+1}\) \((k = 1, \ldots, m)\) is called a conjugate pair.

**Algorithm** (computing a solution to \(Ax = b\)).

**Step 0.** For each \(y \in Y\) set \(x_y = x_1^y - x_2^y, r_y = Ay - b\) and order the pairs \((x_y, r_y)\) in the ordering of \(Y\).

**Step 1.** Set \(j = n\).

**Step 2.** For each conjugate pair \((x_y, r_y), (x_y', r_y')\) in the current sequence compute

\[
\lambda = \begin{cases} 
(r_y')_j/(r_y' - r_y)_j & \text{if } (r_y')_j \neq (r_y)_j \\
1 & \text{otherwise}
\end{cases}
\]

and set

\[
x_y = \lambda x_y + (1 - \lambda)x_y' \\
r_y = \lambda r_y + (1 - \lambda)r_y'.
\]

**Step 3.** Drop out the second half of the sequence.

**Step 4.** If there remains a single pair \((x_y, r_y)\), terminate.

\(x_y\) solves \(Ax = b\) (and \(r_y = 0\)).

**Step 5.** Otherwise set \(j = j - 1\) and go to step 2.

**Proof.** For the purposes of the proof, we shall extend the pairs \((x_y, r_y)\) to quadruples \((x_y, r_y, x_y^1, x_y^2)\), where \(x_y^1, x_y^2\) have their original meaning in step 0 and are updated in step 2 by

\[
x_y^1 = \lambda x_y^1 + (1 - \lambda)x_y^1' \\
x_y^2 = \lambda x_y^2 + (1 - \lambda)x_y^2'.
\]

From this we see that \(x_y = x_y^1 - x_y^2, r_y = Ax_y - b\) hold throughout the algorithm. Below we shall show that each \(\lambda \in [0, 1]\) so that \(x_y^1, x_y^2\) remain nonnegative throughout. We shall prove by induction on \(j = n, \ldots, 1\) that after completing step 2 there always holds...
\begin{align}
(a_y x_{y}^{1} - a_y x_{y}^{2})_i &= (b_y)_i \\
(a_y x_{y}^{1} - b_y) &= (b_y) \\
\lambda_y &= b_y \\
\lambda_y &= b_y
\end{align}

(1 = 1, \ldots, n)

(2 = 1, \ldots, n)

If \( j = n \), then at the beginning of step 2 we have

\begin{align}
\lambda_y &= b_y
\end{align}

for each \( y \in Y \) by assumption; if \( j < n \), then for each \( y \in Y \) corresponding to a quadruple in the current sequence we have at the beginning of step 2

\begin{align}
(a_y x_{y}^{1} - a_y x_{y}^{2})_i &= (b_y)_i \\
(a_y x_{y}^{1} - b_y) &= (b_y)
\end{align}

(1 = 1, \ldots, j)

(1 = j+1, \ldots, n)

due to the inductive assumption. Notice that (b) is a special case of (c) for \( j = n \); thus for each \( j \) we may assume (c) to hold at the beginning of step 2. Since \( y_1 = y_1' \) for each \( i \neq j \) (by ordering), the updated values \( r_y^{1'}, r_y^{2'} \) of \( r_y, r_y^{1}, r_y^{2} \) satisfy

\begin{align}
(a_y x_{y}^{1} - a_y x_{y}^{2})_i &= (b_y)_i \\
(a_y x_{y}^{1} - b_y) &= (b_y)
\end{align}

(1 = 1, \ldots, j-1)

(1 = j+1, \ldots, n).

Since \( y_j = 1, y_j' = -1 \), we have

\begin{align}
(r_y)_j = (a(r_y^{1} - r_y^{2}) - b)_j &\leq (a_y x_{y}^{1} - a_y x_{y}^{2} - b_y)_j = 0 \\
(r_y')_j = (a(r_y^{1} - r_y^{2}) - b)_j &\leq (a_y x_{y}^{1} - a_y x_{y}^{2} - b_y)_j = 0.
\end{align}

If \( (r_y)_j = (r_y')_j \), then from

\begin{align}
\lambda = (a_y x_{y} - b)_j /
\end{align}

we get

\begin{align}
(a_y x_{y} - b)_j = (a_y x_{y} - b)_j
\end{align}

If \( (r_y)_j \neq (r_y')_j \), then both the values are 0 and from \( (a_y x_{y})_j = (a_y x_{y})_j = (a_y x_{y})_j = b_j \) we again obtain (e), which together with (1) gives (a).

Further, \( (r_y)_j > 0 \Rightarrow (r_y)_j \) implies \( \lambda \in \{0, 1\} \), hence \( x_y^{1} > 0, x_y^{2} > 0 \) and \( x_y^{1} \) is a convex combination of \( x_y, x_y^{1} \). This concludes the inductive proof; hence from (a) for \( j = 1 \) we obtain \( a_y x_{y} = b_y \), thus justifying step 4. Since in step 0 we begin with vectors
\[ x_j^2 - x_j^2 \ (j \in Y) \] and at each step 2 a convex combination of two previously computed vectors is taken, the final result must belong to \( \text{Conv} \{ x_j^2 - x_j^2 \ ; \ y \in Y \} \), which completes the proof.

2. Theorems 1 and 2

We shall first prove the lemmas; notice that assertion (i) is generalized here.

**Lemma 1.** Let \( A \) be a regular non matrix and let \( D_j \) be an non matrix whose all rows except the \( j \)-th are zero. Let \( \alpha = 1 + (D_j^{-1})_{jj} \).

Then we have:

(i) \( A + D_j \) is regular if and only if \( \alpha \neq 0 \); in this case,

\[
(A + D_j)^{-1} = A^{-1} - \frac{1}{\alpha}A^{-1}D_jA^{-1},
\]

(ii) if \( \alpha \leq 0 \), then \( A + tD_j \) is singular for some \( t \in (0, 1) \).

**Proof.** Let \( \beta = D_j^{-1} \), then \( A + D_j = (E + \beta)A \) and \( \det(E + \beta) = \alpha \).

Hence \( A + D_j \) is regular iff \( \alpha \neq 0 \). Since \( \beta^2 = (\alpha - 1)I \), we have

\[
(A + D_j)(A^{-1} - \frac{1}{\alpha}A^{-1}D_jA^{-1}) = E - \frac{1}{\alpha} \beta + \beta - \frac{1}{\alpha} \beta^2 = E,
\]

which proves (i). If \( \alpha \leq 0 \), then there is a \( t \in (0, 1) \) with

\[ 1 + t\beta_{jj} = 0. \]

Then \( A + tD_j = (E + t\beta)A \) is singular since

\[ \det(E + t\beta) = 1 + t\beta_{jj} = 0. \]

Before proving theorems 1 and 2, we state this

**Theorem A.** Let \( A^1 \) be regular. Then for each \( A_1, A_2 \in A^1 \), both \( A_1^{-1}A_2^{-1} \) and \( A_1^{-1}A_2^{-1} \) are \( \mathbb{P} \)-matrices.

**Proof.** 1) First we prove that all leading principal minors \( a_1, \ldots, a_n \) of \( A_1A_2^{-1} \) are positive. Put \( D = A_1 - A_2 \), so that \( A_1A_2^{-1} = E + D(A_2^{-1}) \), and let \( D_j \ (j \in N) \) be the matrix whose first \( j \) rows are identical with those of \( D \) and the remaining ones are zero. Then
\[ n_j = \det(B + D^jA_2^{-1}) \text{ for each } j. \text{ We shall prove by induction that } n_j > 0 \quad (j \in \mathbb{N}). \]

1.1) \( j = 1 \): since \( n_1 = \det(B + D^1A_2^{-1}) = 1 + (n_1^{-1}A_2^{-1})_{11} \), the lemma implies \( n_1 > 0 \) for otherwise \( A_2 + tB^1 \) would be singular for some \( t \in (0,1] \), a contradiction.

1.2) Let \( n_{j-1} > 0, \forall \in \mathbb{N}. \) Consider the matrix:
\[
(B + D^jA_2^{-1})(B + D^{j-1}A_2^{-1})^{-1} = (B + (D^j - D^{j-1})A_2^{-1})(B + D^{j-1}A_2^{-1})^{-1}.
\]
Taking determinants on both sides, we obtain:
\[
\frac{n_j}{n_{j-1}} = 1 + ((D^j - D^{j-1})A_2^{-1})(B + D^{j-1}A_2^{-1})^{-1}.
\]
If the right-hand side were nonpositive, then according to lemma 1 the matrix \( A_2 + D^{j-1} + t(D^j - D^{j-1}) = (B + D^{j-1}A_2^{-1} + t(n^j - n^{j-1})A_2^{-1}A_2^{-1} \) would be singular for some \( t \in (0,1] \), a contradiction. Hence:
\[
\frac{n_j}{n_{j-1}} > 0,
\]
so that \( n_j > 0 \) due to the inductive assumption.

2) Second we prove that each principal minor of \( A_2^{-1} \) is positive. Consider a principal minor formed from rows and columns \( k_1, \ldots, k_r \).
Let \( P \) be any permutation matrix with \( P_{i,j} = 1 \) \((i = 1, \ldots, r)\). Then the above minor is equal to the \( r \)-th leading principal minor of:
\[ P_A^{-1}A_2^{-1}P = (P_A^{-1}P)(P_A^{-1}P)^{-1}. \]
Since the interval matrix \( \{P_A P; A \in A \} \) is regular, all leading principal minors of \( (P_A^{-1}P)(P_A^{-1}P)^{-1} \) are positive due to 1).

3) To prove that \( A_2^{-1} \) is also a P-matrix, consider the interval matrix \( (A_2^{-1})^T = \{A^T; A \in A \} \); according to 2), its regularity implies that \( (A_2^{-1})^T(A_2^{-1})^{-1} = (A_1^{-1}A_2)^T \) is a P-matrix, hence so is \( A_1^{-1}A_2 \).
Theorems 1 and 2 are now easy consequences of theorems 0 and 1.

**Theorem 1.** \( A^T \) is regular if and only if \( A_y \) is a P-matrix for each \( y \in Y \).

**Proof.** "Only if." Follows from theorem 1. "If." Take \( y \in Y \), \( j \in N \). Then according to the result by Samelson, Thrall and Wexler [12], the linear complementarity problem \( x^+ = A_y x^- + A_y^{-1} e_j \) has a solution \( x_j \), hence \( A_x x_j^+ - A_y x_j^- = e_j \). Now the regularity follows from theorem 0 since \( A_x x = e_j \) has a solution for each \( A \in A^T \) \( j \in N \).

**Theorem 2.** Let \( A^T \) be regular. Then for each \( y \in Y \), the equation

\[ A_y x_j^+ - A_y x_j^- = b_y \]

has exactly one solution \( x_j \). Moreover, we have \( x_j \in X \) for each \( y \in Y \) and \( \text{Conv} X = \text{Conv} \{ x_j ; y \in Y \} \); especially,

\[ z = \min \{ x_j ; y \in Y \} \]

\[ \overline{z} = \max \{ x_j ; y \in Y \} . \]

**Proof.** From theorem 1 and from the result by Samelson et al. it follows that for each \( y \in Y \) the equation \( x^+ = A_y x^- + w_j \) has exactly one solution \( x_j \), thus satisfying (1). From the equivalent equation \( A_x x_j^+ = b_j \), \( z = \text{sgn} x_j \), we see that \( x_j \in X \). Now, according to theorem 0, for each \( A \in A^T \), \( b \in b^T \) the (unique) solution to \( Ax = b \) belongs to \( X \subseteq \text{Conv} \{ x_j ; y \in Y \} \), hence \( X \subseteq X_1 \) and \( \text{Conv} X \subseteq X_2 \), implying \( \text{Conv} X = X_1 \). So \( z = \min \text{Conv} X = \min \{ x_j ; y \in Y \} \); similarly for \( \overline{z} \).
1. Theorems 3 and 2

Since theorem 5 is a direct consequence of theorem 3, it is placed here just after this theorem, the proof of theorem 4 to be given in the next section. Theorem 3 is proved here in a slightly weaker form.

Theorem 3. Let $A^{2}$ be regular. Then for each $i \in \mathbb{N}$ we have:

(i) $\bar{x}_i = (x^i_y)_i$ for some $y \in Y$ satisfying $(A^{2}y^{i}_y)_i \leq 0$,

where $T_{x_i}x_i \geq 0$,

(ii) $\bar{y}_i = (y^i_y)_i$ for some $y \in Y$ satisfying $(A^{2}y^{i}_y)_i \geq 0$,

where $T_{y_i}y_i \geq 0$.

Proof. We prove (i) only; (ii) is analogous. Let $i \in \mathbb{N}$.

1) First we prove the theorem for the case $\sigma > 0$. Theorem 2 assures the existence of a $y \in Y$ such that $\bar{x}_i = (x^i_y)_i$. Take a $j \in \mathbb{N}$, set $y' = y - 2y_j e_j$ (i.e. $y' = y_j$ and $y'_k = y_k$ for $k \neq j$) and consider the system $A^{2}_{y'}z' = b_{y'}$, $z = 2y_j e_j$. Since $A^{2}_{y'}z' = A^{2}_{y}z' + 2y_j e_j A^{2}_{z'}$ we may use lemma 1 for evaluating $A^{2}_{y'_i}$, which after a lengthy computation gives $z' = \bar{z}_i - \frac{2}{\sigma} (A y_j e_j + \delta) y_j (A^{2}_{y})_j$. Since $\sigma > 0$, $\sigma > 0$ and $\delta \geq \bar{z}_i = (y^i_y)_i$, we obtain $y_j (A^{2}_{y})_j \leq 0$. Since $j$ was arbitrary, we set $(A^{2}_{y})_i \leq 0$.

2) Next let $\sigma > 0$. For $k = 1, 2, \ldots$, let $\bar{\sigma}_k = \sigma + \frac{1}{k} > 0$,

$\bar{b}_k = [c_0 - \bar{\sigma}_k \bar{b}_0 + \bar{c}_k \bar{y}]$ and let $[\bar{x}_k, \bar{x}]$ be the interval solution to $A \bar{x} = \bar{b}_{k}$. According to 1), for each $k$ we have $\bar{x}_k = (x^k_y)_i$ (where $x^k_y$ denotes the vector $x_y$ for the system $A \bar{x} = \bar{b}_{k}$),

$(A^{2}_{y_k} T_{x_k} x_k)_i \leq 0$, $T_{x_k} x_k \geq 0$. Since $Y$ is finite and each $y_k$ belongs to the compact solution set of $A \bar{x} = \bar{b}_{k}$, there exist $y \in Y$, $z \in Y$ and an infinite subsequence $\{k_j\}$ such that $y_{k_j} = y$, $z_{k_j} = z$ for
each \( k_j \) and \( \{x_{yj}^k\} \) is convergent, \( x_{yj}^k \to x \). Since \( A_y x_{yj}^k = b_y + \frac{1}{k_j} T_a y_j, T_a y_j \geq 0 \) for each \( k_j \), taking \( k_j \to \infty \) we obtain \( A_y x = b_y \), \( T_a x \geq 0 \), hence \( x = y \). Since \( (x_{yj}^k)_1 = z_1 + (x_{yj}^k)_1 \to z_1 \) and \( (x_{yj}^k)_2 \to (x_{yj})_2 \), we have \( z_1 = (x_{yj})_1, (x_{yj}^{-1})_2, \leq 0^+ \), \( T_a y_j \geq 0 \).

**Theorem 5.** Under the above [in (5)] notations, we have

\[
\mathbf{E}_1 = \min \left\{(x_{yj})_1 : y \in Y_1 \right\} \quad (1 \in H)
\]

\[
\mathbf{E}_2 = \max \left\{(x_{yj})_2 : y \in Y_2 \right\}.
\]

hence also

\[
\mathbf{E} = \min \{x_{y} : y \in Y_0\}
\]

\[
\mathbf{E} = \max \{x_{y} : y \in Y_0\}.
\]

**Proof.** We shall confine ourselves only to the proof of the formula for \( \mathbf{E}_1 \). According to theorem 3, \( z_1 = (x_{yj})_1 \) for some \( y \in Y \) satisfying \( (x_{yj}^{-1})_1 y_j \leq 0 \) (1 \( y \in H \)). If \( y_j > 0 \), then \( (x_{yj}^{-1})_2 > 0 \), hence \( y_j = -1 \); if \( y_j < 0 \), then \( y_j = 1 \). Thus \( y \in Y_1 \).

Next we prove three unnumbered statements following theorem 5 in (6). Let (5) hold; then for each \( \lambda \in \lambda^+ \), using \( \lambda_0 = \lambda_c - \lambda \), \( |\lambda_0| \leq \lambda \), we may expand \( \lambda^{-1} \) into Neumann series

\[
\lambda^{-1} = (\lambda_c - \lambda_0)^{-1} = \left( \sum_{j=0}^{\infty} (\lambda_c^{-1} \lambda_0)^j \right) \lambda_c^{-1},
\]

implying

\[
|\lambda^{-1} - \lambda_0^{-1}| \leq \left( \sum_{j=0}^{\infty} |\lambda_c^{-1}| \lambda_0^{-1} | \right) = C|\lambda_c^{-1}|.
\]

From this we have \( \lambda^{-1} \in \left[ \lambda_c^{-1} - C|\lambda_c^{-1}|, \lambda_c^{-1} + C|\lambda_c^{-1}| \right] \), an estimation of the interval inverse. Second, if \( 0 < C|\lambda_c^{-1}| < |\lambda_0^{-1}| \), then \( \lambda \) is inverse-stable. Finally we prove that (3) holds for positively (even nonnegative) invertible matrices. Assume for contrary that \( r = g^*(D) = g^*(\lambda_0^{-1} \Delta) \geq 1 \). Then, due to the Perron-Frobenius theorem, \( \lambda_0^{-1} \Delta \Delta x = x \) for some real \( x \neq 0 \), hence \( (\lambda_c - \frac{1}{2} \Delta)x = 0 \), implying singularity; thus \( r \leq 1 \).
4. Theorem 1

We shall prove the finiteness of algorithm 1 in a slightly more general form, proposed by H. Kowarsch, with step 0 being replaced by

Step 0: Select a \( z \in Y \).

(in [9], we set \( z = c \); here, \( z \) is arbitrary).

**Theorem 1.** Let \( A^1 \) be regular. Then the algorithm [with step 0'] is finite for each \( y \in Y \).

**Proof.** Let \( z_0 \) be the initial vector \( z \) in step 0.

1) First assume that \( z_0 = c \). Consider what is going on in the current step of the algorithm. Let \( A_j y_{j}^k = b_j \); put \( x_1^k = \frac{1}{b_j}(b_j - c) \), \( x_2^k = \frac{1}{2}(c - c) \), then \( x_1^k \times k^2 = 0 \), \( x_2^k = x_1^k + x_2^k \) (but, generally, \( x_1^k \) and \( x_2^k \) need not be nonnegative). Then

\[
A_j y_{j}^k = A_j y_{j}^0 - A_j y_{j}^2 = b_j.
\]

Since \( x_1^k = 0 \) \( (z_j = -1) \), \( x_2^k = 0 \) \( (z_j = 1) \), we can see that \( x_1^k \times k^2 \) is a basic solution to the system \( x_1^k = A_j y_{j}^2 + w_j \) with basic variables \( x_1^k \) \( (z_j = 1) \), \( x_2^k \) \( (z_j = -1) \). Moreover, since

\[
k = \min \{ j; z_j x_j^k < 0 \} = \min \{ j; x_1^k > 0 \text{ or } x_2^k < 0 \},
\]

in the next step \( x_1^k \times k^2 \) enters and \( x_2^k \) leaves the basis if \( z_j = -1 \) and conversely if \( z_j = 1 \). Since we started with \( x_1^k = w_j \), \( x_2^k = 0 \), the algorithm in terms of \( x_1^k, x_2^k \) is precisely Dantzig's algorithm [9] for solving the linear complementarity problem \( x^0 = A_j y^0 \times w_j \). This algorithm, as proved in [9], terminates (since \( A_j \) is a P-matrix) in a finite number of steps with \( x_1^k \geq 0, x_2^k \geq 0 \), thus \( x_1^k = x_1^k + x_2^k = 0 \).

2) Let \( z_0 \) be arbitrary. Together with our algorithms, started with \( z_0 \), consider a parallel algorithm applied to the system

\[
\tilde{A}_j \tilde{x}_j = \tilde{b}_j^k, \quad \tilde{y}^k = \begin{bmatrix} A_0 - \Delta_0 & A_0 \tilde{y}_0 + \Delta \end{bmatrix},
\]

for the case \( y \) with the initial vector \( z = c \). We shall prove by induction that at each step...
the current values \( z, \Sigma \) of both algorithms satisfy \( \Sigma = T_2 T_0 \).

This is clear for the initial step, when \( x = 0, \Sigma = s \). Assuming validity in certain step, for the current solutions \( x, \Sigma \) we have

\[
\Sigma = (A_0 T_0 + T_2 \Delta T_0 T_0) \Sigma = A_0 T_0 \Sigma + T_2 \Delta T_0 \Sigma \Sigma = A_0 x = \lambda y \Sigma, \ UserInfo: \text{hence } x = T_0 \Sigma, \]

thus also \( k = \min \{ j ; \Sigma_{j,j} < 0 \} = \min \{ j ; \Sigma_{j,j} < 0 \} \), and the updated values \( x', \Sigma' \) again satisfy \( T_0 \Sigma' = T_0 \Sigma \). Since \( \Sigma \) is regular, the second algorithm terminates due to 1) in a finite number of steps with \( T_2 \Sigma' = 0 \). Then \( T_0 x = T_0 \Sigma' = 0 \), and the first algorithm terminates at the same step.

For a verification of algorithm 2, let \( z' = z - 2 \xi_k e_k \) be the updated vector \( z \). Then \( A_{z'^{+}} = A_{z} + 2 \xi_k \frac{T^{+} \Delta T_{z} e_k}{\xi_k} \), where \( T^{+} \Delta T_{z} e_k \) has all columns except the \( k \)-th zero. We shall use an easily verifiable fact that lemma 1 holds in the same form also in this case if \( \Delta \) is given by \( \Delta = 1 + (A_{z}^{-1} D_{j})_{j,j} \). Then for \( p = 2 \xi_k A_{z}^{-1} T^{+} \Delta e_k + e_k \) and \( y' = A_{z}^{-1} B_{y} \), we have

\[
\begin{align*}
(A_{z}^{-1} y')_{j} & = \frac{1}{\xi_k} (A_{z}^{-1} y')_{k}, \\
(A_{z}^{-1} y')_{j} & = (A_{z}^{-1} y')_{j} - \frac{p_{j}}{\xi_k} (A_{z}^{-1} y')_{k} = (A_{z}^{-1} y')_{j} - p_{j} (A_{z}^{-1} y')_{k} \quad (j \neq k) \\
y'_{j} = \frac{1}{\xi_k} x_{k} \\
x'_{j} = x_{j} - \frac{p_{j}}{\xi_k} x_{k}
\end{align*}
\]

hence pivoting on \( p_{k} \) in the tableau brings \( A_{z}^{-1}, x \) to \( A_{z}^{-1}, x' \).

If \( p_{k} \leq 0 \), then \( A_{z}^{-1} + t (A_{z}^{-1} - A_{z}^{-1}) \) is singular for some \( t \in (0,1) \).

Next we prove the statement in parentheses following (ii). Assume \( D_{j,j} \geq 1 \) for some \( j \notin \Sigma \). Put \( y = -R_{j,j} (A_{0}^{-1})_{j} \), then \( 1 + (A_{0}^{-1} T_{2} \Delta T_{0} \Sigma_{j} j = 1 - D_{j,j} \leq 0 \), hence \( A_{j} + T_{2} \Delta T_{0} \Sigma \Sigma_{j} j \Sigma \) is singular for some \( t \in (0,1) \).
5. Theorem 6-10

Theorem 6. $A^T$ is regular if and only if for each $y \in Y$ the matrix equation

$$B = D_y D^T A_{0}^{-1}$$

has a solution $B_y$. If this condition is met, then $B_y$ is unique for each $y \in Y$. Moreover, for each $A \in A^T$ there exist nonnegative diagonal matrices $L_y$ ($y \in Y$) satisfying $\sum_{y \in Y} L_y = E$ such that

$$A^{-1} = \sum_{y \in Y} B_y L_y$$

holds; especially, we have

$$B = \min\{B_y; y \in Y\}$$

$$\bar{B} = \max\{B_y; y \in Y\}.$$  

Proof. Let $A^T$ be regular. Then, according to theorem 2, for each $y \in Y$ and each $j \in N$ there exists a unique $x_{yj}$ such that

$$A_{yj} x_{yj} = A_{yj} x_{yj} = e_j.$$  

Defining $B_y$ by $(B_y)_{y,j} = x_{yj}$, we obtain

$$A_{yj} B_y = A_{yj} B_y = E,$$

which can be easily rearranged to (g). Conversely, let (g) have a solution $B_y$ for each $y \in Y$. Defining $x_{yj}$ by $x_{yj} = (B_y)_{y,j}$, we have (k). Thus, according to theorem $O, A x = e_j$ has a solution for each $A \in A^T, j \in N$, implying regularity of $A^T$. Furthermore, again from theorem $O, (A^{-1})_{y,j}$, being a solution to $A^T x = e_j$, can be expressed as

$$(A^{-1})_{y,j} = \sum_{y \in Y} \lambda_{yj} x_{yj}$$

with $\lambda_{yj} \geq 0$, $\sum_{y \in Y} \lambda_{yj} = 1$. Now if we define $L_y$ by $(L_y)_{y,j} = \lambda_{yj}$ ($j \in N$) and $(L_y)_{y,i} = 0$ for $i \neq j$, we obtain (h). Finally,

$$d_j = \min \{x_{y,j}; y \in Y\} = \min \{(B_y)_{y,j}; y \in Y\},$$

which gives (j); similarly for $\bar{B}$. 

Theorem 7. Let \( A^T \) be regular and let \( i, j \in \mathbb{N} \). Then, we have:

(i) \( \mathcal{E}_i j = (A^{-1}) y y \mathcal{E}_i j \text{ for some } y, z \in Y \text{ satisfying } (A y y)^{-1}_i \leq 0 \),
\( (A z y y)^{-1}_i j \geq 0 \).

(ii) \( \mathcal{E}_j i = (A^{-1}) y y \mathcal{E}_j i \text{ for some } y, z \in Y \text{ satisfying } (A y y)^{-1}_i j \geq 0 \),
\( (A z y y)^{-1}_i j \leq 0 \).

Proof. Take \( i, j \in \mathbb{N} \). Since \([\mathcal{E}_i j, \mathcal{E}_j i] \) is the interval solution to \( A^T = e_j \), from theorem 3 we get \( \mathcal{E}_i j = (A y y)^{-1}_i j \) for some \( y \) satisfying \( (A y y)^{-1}_i j \leq 0 \), where \( A^T = e_j \), i.e., \( (A z y y)^{-1}_i j \geq 0 \); analogously for \( \mathcal{E}_j i \).

Theorem 8. \( A^T \) is regular if and only if for each \( y \in Y \) and each \( j \in \mathbb{N} \), there exists a \( z \in Y \) such that \( (A z y y)^{-1}_i j \geq 0 \).

Proof. Let \( A^T \) be regular. Then for each \( j \in Y \) and each \( j \in \mathbb{N} \), there exists a \( z \) such that \( (A y y)^{-1}_i j \). Setting \( z = z y y j \), we have \( A y y j = e_j \), hence \( (A z y y)^{-1}_i j \geq 0 \). Conversely, letting \( (A y y)^{-1}_i j \), \( (y y j, j \in \mathbb{N}) \), we see that \( \mathcal{E}_j \) satisfies (6); hence \( A^T \) is regular due to theorem 6.

Unfortunately, theorem 9 is not valid in the form given in [2]; its "only if" part is not true. In order to reformulate it correctly, let us introduce the type of a matrix \( A \) to be a matrix \( Z \) satisfying \( Z_{i j} = 0 \) if \( A_{i j} = 0 \) and \( Z_{i j} = a_{i j} \) otherwise. Then, by definition, \( A^T \) is inverse-stable iff all the matrices \( A^{-1}, A A^{-1} \), have the same type 2, \( |Z| > 0 \).

Theorem 9. \( A^T \) is inverse-stable if and only if all the \( A_{y y}^{-1} \)'s are of the same type 2, \( |Z| > 0 \).

Proof. Only the "if" part is to be proved; put \( x_j = z_j \) (\( j \in \mathbb{N} \)), then from \( (A z y y)^{-1}_i j \), \( y \in Y \), \( j \in \mathbb{N} \) we obtain regularity of \( A^T \) in view of theorem 8. Next, as in the proof of theorem 6, for a given
For some \( \lambda_{y_j} \geq 0 \), \( \sum_{y_j} \lambda_{y_j} y_j = 1 \). Since \( x_{y_j} = (a_j^{-1})_{y_j} \), there holds
\[
T_{y_j} (a_j^{-1})_{y_j} = \sum_{y_j} \lambda_{y_j} T_{y_j} (a_j^{-1})_{y_j} > 0
\]
for each \( j \in \mathcal{J} \), hence \( a_j^{-1} \) is of the type 2.

**Theorem 10.** A regular interval matrix \( a^{-1} \) is positively invertible if and only if \( a_0^{-1} \geq 0 \).

**Proof.** By assumption, \( (a_0 + \Delta)^{-1} \geq 0 \). In the light of the well-known matrix theorem, it will suffice to show that \( (a_0 - \Delta)^{-1} > 0 \).

For each \( j = 0, 1, \ldots, n \), define \( a_j \in a^{-1} \) by
\[
(a_j)_{i,j} = \begin{cases} (a_0 - \Delta)_{i,j} & (i = 1, \ldots, j) \\ (a_0 + \Delta)_{i,j} & (i = j+1, \ldots, n) \\ \end{cases}
\]

We shall prove by induction that \( a_j^{-1} \geq 0 \) for each \( j \). Since \( a_j = a_0 + \Delta_j \), the first step follows from the assumption. Now let \( a_{j-1}^{-1} \geq 0 \) (for \( j \in \mathbb{N} \)).

Let \( D_j = a_j - a_{j-1} \), then all the rows of \( D_j \) are zero except the \( j \)-th which is equal to \( -2 \Delta_j \). Hence lemmas 1 gives
\[
A_j^{-1} = (a_{j-1} + D_j)^{-1} = a_{j-1}^{-1} - \frac{a_{j-1}^{-1} D_j a_{j-1}^{-1}}{\Delta_j}
\]
Since \( a_{j-1}^{-1} \geq 0 \), \( D_j \leq 0 \) and \( \Delta_j > 0 \), we have \( a_j^{-1} \geq 0 \), concluding the induction. Hence \( (a_0 - \Delta)^{-1} = a_n^{-1} \geq 0 \).

Notice that the proof goes through also for nonnegatively invertible interval matrices \( a^{-1} \geq 0 \) for each \( a \in a^{-1} \) if the condition is changed to "\( a_0^{-1} \geq 0 \)."
References:


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