A NOTE ON SOLVING EQUATIONS OF TYPE $A^I x^I = b^I$

by

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Let $A^I = \begin{bmatrix} A_0 - \Delta & A_0 + \Delta \end{bmatrix}$ be a regular $n \times n$ interval matrix and let $b^I = \begin{bmatrix} b_1 & b_2 \end{bmatrix}$ be an interval $n$-vector. In this note we show that the problem of finding an interval $n$-vector $x^I$ such that $A^I x^I = b^I$ (where the left-hand multiplication is performed in interval arithmetic) can be rather easily solved if we impose an additional restriction on the concept of solution.

**Definition.** An interval $n$-vector $x^I$ is called a strong solution if $A^I x^I = b^I$ and, moreover, if there exist $x_1, x_2 \in x^I$ such that $A_1 x_1 = b_1$, $A_2 x_2 = b_2$ for some $A_1, A_2 \in A^I$.

We shall show that the problem of finding a strong solution or verifying that no such solution exists can be solved by the following simple algorithm:

**Algorithm.**

0. Solve the equations $A_0 x_1 - \Delta |x_1| = b_1$, $A_0 x_2 + \Delta |x_2| = b_2$.

1. Construct $\mathcal{X}^I = \{x_1, x_2\}$, where $x_j = \min\{x_1, x_2\}$,

   $x_j = \max\{x_1, x_2\}$, $j = 1, \ldots, n$.

2. If $A^I \mathcal{X}^I = b^I$, stop! $\mathcal{X}^I$ is a strong solution.

3. Otherwise stop! No strong solution exists.

Since $A^I$ is regular, each of the two equations described in step 0 has a unique solution, as proved in [4]. Since $|x_i| = T_z x_i$ for some diagonal matrix $T_z$ satisfying $|T_z| = E$, we have $(A_0 - \Delta T_z) x_1 = b_1$, where $A_0 - \Delta T_z \in A^I$.

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similarly for \( x_2 \). Hence if \( A^I x^I = b^I \), then \( x^I \) is a strong solution (since \( x_1, x_2 \in \mathcal{Z}^I \)). To justify step 3, we prove this result:

**Theorem.** Let \( A^I \) be regular and let \( A^I x^I = b^I \) have a strong solution. Then \( x^I \) is also a strong solution.

**Proof.** Let \( x^I \) be a strong solution. Then \( A_1 x_1^* = b \), \( A_2 x_2^* = b \) for some \( x_1^* \), \( x_2^* \in \mathcal{Z}^I \), \( A_1, A_2 \in \mathcal{A}^I \). Due to the Ottiti-Prager theorem, we have \( \{ A x^I : A \in \mathcal{A}^I \} = \{ A_1 x_1^* - A_2 x_2^* \} \) implies \( A_1 x_1^* = A_2 x_2^* = b \) and the above-mentioned uniqueness of solution gives \( x_1^* = x_1 \). In a similar way we obtain \( x_2^* = x_2 \); hence \( x^I \subset x^I \). Now we have \( b^I \subset A^I x^I \subset A^I x^I = b^I \), \( b = A_1 x_1 \), \( b = A_2 x_2 \), hence \( x^I \) is a strong solution.

We shall briefly sum up some methods for solving the equation \( A_0 x_1^* - A|x_1^*| = b \) (similarly for \( A_0 x_2 + A|x_2| = b \)). As described in [3], we have these options:

(a) to solve the linear complementarity problem
\[
x_1^* = (A_0 - \Delta)^{-1}((A_0 + \Delta)x_1^* + (A_0 - \Delta)^{-1}b)
\]
(b) to solve the system \((A_0 - \Delta T_z)x = b\) until \( T_z x \geq 0 \); if \( T_z x \) is not nonnegative in the current step, we set \( z_k = -z_k \), where \( k = \min \{ j : z_j x_j \leq 0 \} \) and return \((T_z \text{ is a diagonal matrix with diagonal elements } z_1^*, \ldots, z_n^*)\);
(c) to solve the fixed-point equation \( x_1 = A_0^{-1} \Delta |x_1^*| + A_0^{-1} b \) by Banach iterations \( x^{m+1} = A_0^{-1} \Delta |x^m| + \)


\[ a \sigma a \quad \text{and} \quad a \sigma b \quad \Rightarrow \quad a \sigma c \]

+ \[ a^{-1} b \quad \text{we have} \quad x^* \rightarrow x \quad \text{provided} \]
\[ J(a^{-1} d) < 1. \]

**Example 1** (Hansen [2]). The system
\[
\begin{align*}
(2,3) x_1 + (0,1) x_2 &= (0, 120) \\
(1,2) x_1 + (2,3) x_2 &= (60, 240)
\end{align*}
\]

has a unique strong solution \( x^T = [5, 3] \), where \( x = (0, 17, 1429)^T \).

**Example 2** (Barth, Nuding [1]). The system
\[
\begin{align*}
(2,4) x_1 + (1,2) x_2 &= (5, 22) \\
(-1,2) x_1 + (2,4) x_2 &= (-2, 2)
\end{align*}
\]

has no strong solution.

**References**


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