TESTING REGULARITY OF INTERVAL MATRICES

by

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An \( n \times n \) interval matrix \( A^I = [A_o - \Delta, A_o + \Delta] \) is called regular if \( \det A \neq 0 \) for each \( A \in A^I \), otherwise it is said to be singular. Testing regularity may be a difficult task in the general case. In fact, all the necessary and sufficient regularity conditions arrived at by the author ([3],[5],[6]) are of the form "\( A^I \) is regular if and only if \( \forall y \in Y \ldots " \), where \( Y = \{y \in \mathbb{R}^n : |y_j| = 1 \forall j\} \), so that in case of regularity they require performing \( 2^n \) computations of some sort (testing singularity is generally easier because it suffices to find some \( y \in Y \) for which the respective condition is violated). Therefore it is reasonable to try some simpler sufficient regularity (singularity) conditions first, resorting to necessary and sufficient conditions only if the previous tests fail. According to our computational experience, we recommend the following sequence of tests:

0. Compute \( D = |A_o^{-1}| \Delta \).
1. If \( D_{jj} \geq 1 \) for some \( j \), then \( A^I \) is singular.
2. If \( \sigma(D) < 1 \), then \( A^I \) is regular.

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4. Solve the system \( A_{yz} x = y \), \( T_z x \not\geq 0 \) for different \( y \in Y \) using the sign - accord algorithm \( [3, p.6] \).

If cycling occurs for some \( y \in Y \), then \( A^T \) is singular.

5. Otherwise (i.e. if \( A_{yz} x = y \), \( T_z x \not\geq 0 \) has a solution for each \( y \in Y \)), \( A^T \) is regular.

For a justification of step 1, see \( [3, p.8] \) and \( [4, p.26] \).

Step 2 is the well-known Beeck's spectral radius criterion \( [2] \). Algorithm 5.1 \( [6, p.112] \) will perform better if \( j \)

is always chosen so that \( k_j \neq 0 \) and \( \sum_k (A_c - A)^{-1} j_k \)

is minimal. Steps 4 and 5 are based on the assertion \( \text{(iii)} \)

of theorem 4.1 in \( [6] \); as before, we use the notations

\( T_z = \text{diag}\{z_1, \ldots, z_n\} \), \( A_{yz} = A_c - T_y \Delta T_z \). We recommend

to start the sign - accord algorithm with \( z = \text{sgn}(A_c^{-1} y) \),

as mentioned in \( [5, p.41] \), since this reduces considerably

the number of systems to be solved. According to theorem 4

in \( [3, p.7] \), in case of regularity the sign - accord algo-

rithm finds for each \( y \in Y \) the (unique) solution to

\( A_{yz} x = y \), \( T_z x \not\geq 0 \) in a finite number of steps. If \( A^T \) is

singular, then the latter system does not have a solution

for some \( y \in Y \), hence the sign - accord algorithm either

cycles (returns to the same \( z \) and \( x \) after several steps),

or finds a singular matrix \( A_{yz} \).

We add five examples. For each \( i \), the \( i \)-th example

terminates in the \( i \)-th step (\( i=1, \ldots, 5 \)).
Example 1.
\[
\begin{pmatrix}
[1, 2] & [3, 4] \\
[2, 5] & [7, 8]
\end{pmatrix}
\]

We have \( \max_j D_{jj} = 9 \) ; singular matrix.

Example 2 (Barth, Nuding [1, p.118]).
\[
\begin{pmatrix}
[2, 4] & [-2, 4] \\
[-1, 2] & [2, 4]
\end{pmatrix}
\]

Here, \( \max_j D_{jj} = 0.405 \) and \( \Phi(D) = 0.946 < 1 \) ; regular matrix.

Example 3.
\[
\begin{pmatrix}
[1, 2] & [3, 4] \\
[5, 6] & [7, 8]
\end{pmatrix}
\]

Since \( \max_j D_{jj} = 0.687 \) and \( \Phi(D) = 1.125 \), we use the descent algorithm (step 3) which after two iterations produces a singular matrix
\[
\begin{pmatrix}
2 & 3 \\
5 & 7.5
\end{pmatrix}
\]

Example 4 (Baumann [6, p.113]).
\[
\begin{pmatrix}
[0, 2] & [2, 4] \\
[1, 3] & [0, 2]
\end{pmatrix}
\]
Here, $\max_j D_{jj} = 0.8$, $\Omega(D) = 1.4$ and the descent algorithm fails. Using the sign-accord algorithm for $y = (-1,1)$, we obtain this sequence of $z$'s and $x$'s:

\[
\begin{array}{cc}
z & x \\
(1, -1) & (-2, 1.5) \\
(-1, -1) & (0.667, -0.5) \\
(1, -1) & (-2, 1.5)
\end{array}
\]

hence the algorithm cycles; singular matrix.

**Example 8 (Baumann [7])**,

\[
\begin{pmatrix}
0.25 & 1.75 \\
-1.75 & -0.25
\end{pmatrix} \begin{pmatrix}
0.25 & 1.75 \\
0.25 & 1.75
\end{pmatrix}
\]

For this example, $\max_j D_{jj} = 0.75$, $\Omega(D) = 1.5$ and the descent algorithm fails. However, for each $y \in Y$ the sign-accord algorithm finds a solution $x_y$ to $A_y z = y$, $T_z x \preceq 0$:

\[
\begin{array}{cc}
y & x_y \\
(1, 1) & (0, 4) \\
(1, -1) & (4, 0) \\
(-1, 1) & (-4, 0) \\
(-1, -1) & (0, -4)
\end{array}
\]

Regular matrix.
References


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