A NOTE ON THE SIGN-ACCORD ALGORITHM

by

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In our papers previously published in Freiburger Intervall-Berichte [1], [2], [3] we showed that each vertex $x_y$ of the convex hull of the solution set of an interval linear system $A^I x = b^I$ with regular interval matrix $A^I$ can be described as a unique solution of the system

$$
A_{yz}^I x = b_y^I,
T_z^I x \geq 0
$$

(here, $y,z \in \mathbb{Y} = \{y_j \; | \; \gamma_j = 1 \; \forall j\}$, $T_z = \text{diag}\{z_1, \ldots, z_n\}$, $A^I_{yz} = A_c^I - T_y A^I T_z$, $b_y^I = b_c^I + T_y \delta$, where $A^I = [A_c^I - \delta, A_c^I + \delta]$, $b^I = [b_c^I - \delta, b_c^I + \delta]$) and we proposed the following finite algorithm (called the sign-accord one since it works toward reaching $x_j^I x_j^I \geq 0 \; \forall j$) for solving (1) [1, p.6], [2, p.25]:

0. Select a $z \in \mathbb{Y}$.
1. Solve $A_{yz}^I x = b_y^I$.
2. If $T_z^I x \geq 0$, stop with $x = x$.
3. Otherwise find $k = \min \{j \; | \; z_j x_j < 0\}$.
4. Set $x_k = -x_k$ and go to step 1.

Later [3, p.31] we recommended to specify step 0 by

$0^0$. Set $z = \text{sgn}(A_c^{-1} b_y)$

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(where \( (\text{sgn} \; x)_i = 1 \) if \( x_i \geq 0 \) and \( (\text{sgn} \; x)_i = -1 \) otherwise). The idea behind it was quite simple: replacing the equation \( A_{yz} x = b_y \) (with unknown \( z \)) by \( A_c x' = b_y \), one may expect its solution \( x' = A_c^{-1} b_y \) to lie in the same orthant as \( x_y \) provided \( A_c^{-1} \) is "narrow". Our recent computational experience confirmed the impact of step 0° upon the behavior of the algorithm, resulting in most cases in going through step 1 only once; this, in fact, was the main reason for writing this note. We shall first support our above - stated intuitive reasoning by some theoretical result and then we shall show an example of a worst-case behavior caused by an improper initialization, where the application of step 0° leads to a drastic reduction of the number of systems to be solved.

Let \( D = |A_c^{-1}| \). We have this result:

**Theorem 1.** Let \( D|x_y| < |x_y| \) for some \( y \in Y \). Then the sign-accord algorithm with step 0° finds \( x_y \) in only one iteration.

**Proof.** Since \( |x_y| > 0 \), there exists a unique \( z \in Y \) (namely, \( z = \text{sgn} \; x_y \)) such that \( A_{yz} x_y = b_y \), \( T_z x_y \geq 0 \) holds. Denote \( x' = A_c^{-1} b_y \). Then from \( A_c x_y = T_y A_c x_y + b_y = T_y \Delta|x_y| + b_y \), \( A_c x' = b_y \) we obtain \( A_c(x_y - x') = T_y \Delta|x_y| \), implying \( |x_y - x'| \leq D|x_y| \). Hence \( x_y \) and \( x' \) lie in the same orthant, so that \( z = \text{sgn} \; x' \). Since the sign-accord algorithm starts in step 0° with \( z = \text{sgn} \; x' \), the solution to \( A_{yz} x = b_y \) found in step 1 is identical with \( x_y \), so that \( T_z x \geq 0 \) in step 2 and the algorithm stops.
Since \( D \rightarrow 0 \) as \( A \rightarrow 0 \), the condition \( D|x_y| < |x_y| \) is satisfied if \( |x_y| > 0 \) and \( A \) is sufficiently narrow.

Now, for each \( n \geq 2 \) consider the interval linear system

\[
A_n x = [-e, e]
\]

(2)

where \( e = (1, 1, \ldots, 1) \in \mathbb{R}^n \) and the \( n \times n \) interval matrix \( A_n \) is defined by

\[
(A_n)_{i,j} = \begin{cases} 
1 & \text{if } i = j \\
[-2, 2] & \text{if } j = i + 1 \text{ and } 1 \leq i \leq n - 1 \\
0 & \text{otherwise}
\end{cases}
\]

(it differs only in the right-hand side term from the example (3.1) studied in [3, p.40]).

**Theorem 2.** Let \( n \geq 2 \). Then, for the interval linear system (2), we have:

(i) for each \( y \in Y \), the sign-accord algorithm, when started from \( z = (y_1, y_2, \ldots, y_{n-1}, -y_n) \) in step 0, solves \( 2^n \) systems to find \( x_y \);

(ii) for each \( y \in Y \), the sign-accord algorithm, starting with step \( 0^0 \), solves only one system to find \( x_y \).

**Proof.** First we find by backward substitutions that for each \( y, z \in Y \) the solution of the system \( A y x = b_y \) is given by

\[
x_j = y_j \sum_{m=0}^{n-j} 2^{j+m} \prod_{i=j+1}^{i=m} y_i z_i \quad (j = 1, \ldots, n)
\]

(where we employ the usual convention \( \sum_{\emptyset} = 0, \prod_{\emptyset} = 1 \)). Hence
\[ z_j x_j = \sum_{m=0}^{n-j} z^m \prod_{i=j}^{j+m} y_i x_i \quad (j = 1, \ldots, n) \]

and since the last term prevails, we have

\[ \text{sgn}(z_j x_j) = \text{sgn}\left( \prod_{i=j}^{n} y_i x_i \right) = \prod_{i=j}^{n} y_i x_i \quad \text{for each } j = 1, \ldots, n. \]

Next we prove that for each \( y \in Y \), the number \( p_y(z) \) of systems the sign-accord algorithm must solve to find \( x_y \) when started from vector \( z \) in step 0 is given by

\[ p_y(z) = 1 + \sum_{j=1}^{n} \left( 1 - \prod_{i=j}^{n} y_i x_i \right) z^{j-2}. \quad (3) \]

We shall carry out the proof by induction on \( p_y(z) \). If \( p_y(z) = 1 \), then the sign-accord algorithm, after solving

\[ A_y z = b_y \quad \text{stops with } T_x \leq 0. \]

Hence for each \( j \) we have \( \prod_{i=j}^{n} y_i x_i = \text{sgn}(z_j x_j) = 1 \), so that the right-hand side in (3) is equal to 1. Now assume that (3) holds for each \( y, z \) with \( p_y(z) \leq r \) and let \( y, z \) be such that \( p_y(z) = r+1 \).

Let \( z' \) be the updated value of \( z \) after passing for the first time through step 4. Then \( z'_k = -z_k \), \( z'_j = z_j \) for \( j \neq k \), \( \prod_{i=k}^{n} y_i x_i = \text{sgn}(z_j x_j) = 1 \) for \( j < k \), \( \prod_{i=j}^{n} y_i x_i = \text{sgn}(z_j x_j) = -1 \), hence by the inductive assumption,

\[ p_y(z) = 1 + p_y(z') = 2 + \sum_{j=1}^{r} \left( 1 - \prod_{i=j}^{n} y_i x_i \right) z^{j-2} = \ldots = \]

\[ = 1 + \sum_{j=1}^{n} \left( 1 - \prod_{i=j}^{n} y_i x_i \right) z^{j-2} \quad \text{(since } \prod_{j}^{j} y_i x_i = - \prod_{j}^{j} y_i x_i \text{ for } j > k \text{), which}

completes the inductive proof of (3).
Now, if \( z = (y_1, y_2, \ldots, y_{n-1}, y_n) \), then \( \prod_{j} y_i z_i = -1 \)
for each \( j \), hence \( p_y(z) = 1 + \sum_{j=1}^{n} 2^{j-1} = 2^n \), which
proves (i). Using step 0°, we have \( z = \text{sgn}(\lambda_0^{-1} b_y) = y \)
(since \( \lambda_0 = E \) and \( b_y = y \)), hence \( \prod_{j} y_i z_i = 1 \) for
each \( j \), implying \( p_y(z) = 1 \) in this case, which completes
the proof.

**Remark.** The equation (3) has also another interesting
consequences. E.g., for each \( y \in Y \) and each \( k, 1 \leq k \leq 2^n \),
there exists a \( z \in Y \) such that \( p_y(z) = k \), etc.

**References**

Intervall-Berichte 84/7, 1-14

Freiburger Intervall-Berichte 84/7, 17-30

[3] J. Rohn, Interval Linear Systems, Freiburger Intervall-
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