A vector $x \in \mathbb{R}^n$ is called an inner solution of a system of linear interval equations $A^T x = b^T (A^T = [A_1, A_2], A_c = [a_c - \Delta, a_c + \Delta])$ of size $m \times n$, $b^T = [b_1, b_2] = [0_c^- \Delta, b_c + \Delta])$, if $Ax \in b^T$ for each $A \in A^T$ (for a motivation, see [11]). Denote by $X_i$ the set of all inner solutions. We have this characterization:

Theorem. $x \in X_1$ if and only if $x = x_1 - x_2$, where $x_1, x_2$ is a solution to the system of linear inequalities

$A x_1 - Ax_2 \leq b$

$-Ax_1 + Ax_2 \leq -b$

$x_1 \geq 0, x_2 \geq 0$.  

(S)

Proof. Due to Oettli-Prager theorem, $\{Ax; A \in A^T\} = [A_c x - \Delta |x|, A_c x + \Delta |x|]$. "Only if": Let $x \in X_1$, then $b \leq A_c x - \Delta |x|$ and $A_c x + \Delta |x| \leq b$, substituting $x = x^+ - x^-$, $|x| = x^+ + x^-$, we see that $x_1 = x^+$, $x_2 = x^-$ satisfy (S). "If": Let $x_1, x_2$ solve (S); define $d \in \mathbb{R}^n$ by $d_j = \min \{x_{1j}, x_{2j}\}$, then $d \geq 0$ and for $x = x_1 - x_2$ we have $x = x^+ - x^-$, hence $A_c x + \Delta |x| = Ax_1 - Ax_2 - 2 \Delta d \leq b$, similarly $A_c x - \Delta |x| \geq b$. Thus $[A_c x - \Delta |x|, A_c x + \Delta |x|] \subseteq b^T$, implying $x \in X_1$.

As consequences, we obtain: (i) $X_1$ is a convex polytope, (ii) each $x \in X_1$ satisfies $\Delta |x| \leq \delta$ (by adding the first two inequalities in (S)), (iii) $X_1$ is bounded if for each $j$ there is a $k$ with $\alpha_{kj} > 0$ (since then from (ii) follows $|x_j| \leq \alpha_j / \alpha_{kj}$), (iv) $X_1 \neq \emptyset$ if and only if (S) has a solution, which can be tested by phase I of the simplex algorithm, (v) for $x_j = \min \{x_{1j}, x_{2j}\}$ we have $x_j = \min \{x_{1j} - x_{2j}\}$, $x_1, x_2$ solve (S), which is a linear programming problem (similarly for $x_j = \max \ldots$), (vi) nonnegative inner solutions are described by $Ax \leq b$, $-Ax \leq -b$, $x \geq 0$, (vii) also, $X_1 = \{x; Ax - b \leq -\Delta |x| + \delta\}$ (observe the similarity with the Oettli-Prager result).

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Reference