EIGENVALUES OF A SYMMETRIC INTERVAL MATRIX

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Abstract. Exact bounds for eigenvalues of a symmetric interval matrix of the form $\mathbf{A} = (A - rr^T, A + rr^T)$ (\(A\) symmetric, \(r > 0\)) are given under assumptions that all eigenvalues of \(A\) are mutually different, the eigenvectors of \(A\) have nonzero entries and \(r\) is sufficiently small in norm to preserve these properties over \(A^2\).

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In this paper we investigate the eigenvalues of a symmetric interval matrix $\mathbf{A} = [A_0 - rr^T, A_0 + rr^T]$, where \(A_0\) is a symmetric \(n \times n\) matrix and \(r\) is a (column) vector whose all entries are positive. We shall give the results under three assumptions. First we shall assume that

(i) each \(A \in \mathbf{A}\) has \(n\) different real eigenvalues

\[ \lambda_1(A) < \lambda_2(A) < \ldots < \lambda_n(A). \]

Then we may define the sets

\[ L_i = \{ \lambda_i(A); A \in \mathbf{A} \}, \quad (i = 1, \ldots, n). \]

Second we shall assume that

(ii) \(L_i \cap L_j = \emptyset\) for \(i \neq j\) \((i, j = 1, \ldots, n)\)

holds. Before formulating the third assumption, we first introduce, for any \(a \in \mathbb{R}^n\), the matrix \(T_a\) as the diagonal matrix with diagonal vector \(a\), and define \(Y = \{ z \in \mathbb{R}^n; |z_j| = 1\text{ for each }j\}\). We assume

(iii) for each \(i \in \{1, \ldots, n\}\) there exists a \(y_i \in Y\) such that each eigenvector \(x\) corresponding to an eigenvalue from \(L_i\) satisfies either \(T_{y_i}x > 0\), or \(T_{y_i}x < 0\).
Here the inequalities are to be understood componentwise.
If we introduce the signature vector $\text{sgn } x$ of a vector $x \in \mathbb{R}^n$
by $(\text{sgn } x)_i = 1$ if $x_i > 0$ and $(\text{sgn } x)_i = -1$ otherwise, then each
eigenvector corresponding to an eigenvalue from $L_i$ satisfies
$\text{sgn } x = y_i$ or $\text{sgn } x = -y_i$. To simplify notations, denote
$T_i = T_i$. Since eigenvectors corresponding to different eigenvalues of $A_0$ are orthogonal, we have $y_i \neq y_j$, thus also $T_i \neq T_j$,
for each $i \neq j$.

In the key part of the proof of Theorem 1, we shall use
the following lemma, which is of independent interest.

**Lemma.** Let $B$ be a regular $n \times n$ matrix and let $p, q$ be non-
negative vectors from $\mathbb{R}^n$. Then the interval matrix
$[B - qp^T, B + qp^T]$ is singular if and only if
$z^T p B^{-1} T_q y \geq 1$
holds for some $z, y \in Y$.

**Proof.** According to Theorem 6.3 in [2, p. 44],
$[B - qp^T, B + qp^T]$ is singular if and only if there exist
$z, y \in Y$ such that the matrix $B^{-1} T_q q B^T y$ has a real eigenvalue $\lambda$
with $|\lambda| \geq 1$. Then $B^{-1} T_q q B^T x = (p^T x) B^{-1} T_q y = \lambda x$ for some
$x \neq 0$, where $p^T x \neq 0$ due to $\lambda \neq 0$, hence premultiplying the
equation by $p^T y$ gives $p^T y B^{-1} T_q y = \lambda$. Setting $z = -z$ if
$\lambda < 0$, we obtain $z^T B^{-1} T_q y = p^T y B^{-1} T_q y = |\lambda| \geq 1$.

In the main theorem to follow, we give exact bounds for
eigenvalues and also prove that the extremal eigenvalues are
achieved at some symmetric matrices from $\Lambda^T$.

Theorem 1. Let $r > 0$ and let (i), (ii), (iii) hold. Then for each $i \in \{1, \ldots, n\}$ we have

$$L_i = [\lambda_i, \overline{\lambda}_i]$$

where

$$\lambda_i = \min \left\{ \lambda_i(A_0 - D_i), \lambda_i(A_0 + D_i) \right\}$$

$$\overline{\lambda}_i = \max \left\{ \lambda_i(A_0 - D_i), \lambda_i(A_0 + D_i) \right\}$$

and

$$D_i = T_i x^T T_i.$$ 

Proof. The proof consists of several steps. Let $i \in \{1, \ldots, n\}$.

(a) We prove that $L_i$ is compact. If $\lambda \in L_i$, then $\lambda = x^T A x$ for some $A \in \mathbb{R}^n$ and $x$ satisfying $\|x\|_2 = 1$, hence $L_i$ is bounded. To prove that $L_i$ is closed, let $\lambda_j \in L_i$ ($j = 1, 2, \ldots$) and $\lambda_j \to \lambda$. Then $\lambda_j^2 x_j^2 = \lambda_j^2 x_j^2$ for some $A_j \in \mathbb{R}^n$, $\|x_j\|_2 = 1$, $T_j x_j \geq 0$ ($j = 1, 2, \ldots$) and there exists a subsequence $\{j_k\}$ such that $A_{j_k} \to A \in \mathbb{R}^n$, $x_{j_k} \to x$, $\|x\|_2 = 1$, $T_i x \geq 0$, $Ax = \lambda x$. Since $x$ is an eigenvector, it must be $T_i x \geq 0$ due to (iii), thus $x$ corresponds to an eigenvalue from $L_i$; this shows that $\lambda \in L_i$ so that $L_i$ is closed and thus also compact.

(b) Next we show that $\lambda_i(A_0) \in L_i^0$, the interior of $L_i$. Take an eigenvector $x$ of $A_0$ corresponding to $\lambda_i(A_0)$ and choose an $\varepsilon_0 > 0$ such that $\sqrt{\varepsilon_0} |x| \leq r$ and $(\lambda_i(A_0) + \varepsilon_0) x^2 2 = \lambda_i(A_0) + \varepsilon_0 x^2 2$, $(\lambda_i(A_0) + \varepsilon_0 x^2 2) x, \lambda_i(A_0) + \varepsilon_0 x^2 2, A_0 + \varepsilon xx^T x, \lambda_i(A_0) + \varepsilon xx^T x,$ and $(A_0 + \varepsilon xx^T) x = \lambda_i(A_0) + \varepsilon xx^T x,$ hence $\lambda_i(A_0) + \varepsilon xx^T x$ is an eigenvalue from $L_i$; thus $\lambda_i(A_0) \in L_i^0$.

(c) In view of (a), $L_i - L_i^0 \neq \emptyset$. Let $\lambda \in L_i - L_i^0$. We shall prove that either $\lambda = \lambda_i(A_0 - D_i)$, or $\lambda = \lambda_i(A_0 + D_i)$. Since the
interval matrix \[ [A_0 - \lambda E - xx^T, A_0 - \lambda E + xx^T] \]
is singular and \( \lambda \) is not an eigenvalue of \( A_0 \) in view of (b) and (ii), the lemma above guarantees the existence of \( z, y \in Y \) such that \( x^T y (A_0 - \lambda E)^{-1} T_y y \geq 1 \). Assume for contrary that
\[ x^T y (A_0 - \lambda E)^{-1} T_y y > 1. \]
Then there exists an \( \epsilon_1 > 0 \) such that
\[ (\lambda - \epsilon_1, \lambda + \epsilon_1) \cap L_j = \emptyset \]
for each \( j \neq i \) and \( x^T y (A_0 - \lambda E)^{-1} T_y y > 1 \)
for each \( \lambda \in (\lambda - \epsilon_1, \lambda + \epsilon_1) \), which, again employing the lemma, gives that \( (\lambda - \epsilon_1, \lambda + \epsilon_1) \subset L_4 \) contrary to \( \lambda \notin L_4 \).

Hence
\[ x^T y (A_0 - \lambda E)^{-1} T_y y = 1 \]
holds. Put \( x = (A_0 - \lambda E)^{-1} T_y y \) and \( p = (A_0 - \lambda E)^{-1} T_y x \), then
\[ x^T y = y^T x = 1 \] and \( (A_0 - T_y y z y^T x) x = A_0 x - T_y y = \lambda x \),
\( (A_0 - T_y y z y^T x) p = \lambda p \), hence \( x \) and \( p \) are eigenvectors corresponding to \( \lambda \) (since \( T_y y z y^T x = x^T \); implying \( A_0 - T_y y z y^T x = A^T \)).
Similarly \( A_0 - T_y y z y^T x \leq A^T \). We shall prove that \( z_j x_j > 0 \) for each \( j \). In fact, assuming \( z_j x_j < 0 \) for some \( j \) (the possibility of \( z_j x_j = 0 \) is precluded by (iii)), for \( z_j Y \) given by \( z_j = -z_j \) and \( z_k = z_k \) for \( k \neq j \) we would have \( z^T y (A_0 - \lambda E)^{-1} T_y y = z^T y x \)
\( > x^T y = 1 \) contrary to \( \lambda \notin L_4 \), as before. Hence \( z = sgn x = -y_4 \)
and in a similar way, \( y = sgn p = -y_4 \). Since, as established above, \( \lambda \) is an eigenvalue of \( A_0 - T_y y z y^T x \), there holds either
\[ \lambda = \lambda_4 (A_0 - T_y y z y^T x) = \lambda_4 (A_0 - D_4), \]
or
\[ \lambda = \lambda_4 (A_0 + T_y y z y^T x) = \lambda_4 (A_0 + D_4). \]

(a) We have proved that \( L_4 \) is a compact set with nonempty interior and (at most) two boundary points. Hence \( L_4 = [\lambda_4, \lambda_4] \),
where \( \lambda_4, \lambda_4 \) are the two boundary points, satisfying (1) in view of (c), and both \( A_0 - D_4 \) and \( A_0 + D_4 \) are symmetric.

Next we prove that each \( \lambda \in L_4 \) is an eigenvalue of a matrix
in some special form:

**Theorem 2.** Let \( r > 0 \) and let (i), (ii), (iii) hold. Then for each \( \lambda \in L_1, i \in \{1, \ldots, n\} \), there exists a \( t \in [-1, 1] \) such that \( \lambda = \lambda_i(A_0 + tD_4) \).

**Proof.** The assertion obviously holds for \( \lambda = \lambda_i(A_0) \) with \( t = 0 \). If \( \lambda \in L_1, \lambda \neq \lambda_i(A_0) \), then \( z^T T_x(A_0 - \lambda E)^{-1} T_x y_i \geq 1 \) for some \( z, y_i \in Y \). Hence if \( z, y \in Y \) satisfy

\[
 z^T T_x(A_0 - \lambda E)^{-1} T_x y = \max \left\{ z^T T_x(A_0 - \lambda E)^{-1} T_x z : z, y \in Y \right\}
\]

then for \( x = (A_0 - \lambda E)^{-1} T_x y, p = (A_0 - \lambda E)^{-1} T_x z \) we obtain, in a similar way as in the part (c) of the above proof,

\[
 z^T T_x x = y^T T_x p \geq 1
\]

\[
 (A_0 - \frac{y T_x z^T}{z^T T_x x}) x = \lambda x
\]

\[
 (A_0 - \frac{y T_x z^T}{y^T T_x p}) p = \lambda p
\]

and the optimality of \( z, y \) gives \( z = \text{sgn} x = \pm y_i, y = \text{sgn} p = \pm y_i \) implying \( \lambda = \lambda_i(A_0 + tD_4) \) where \( t = \pm \frac{1}{z^T T_x x} \), so that \( t \in [-1, 1] \). \( \square \)

Finally we show that for each \( \lambda \in L_1, (i = 1, \ldots, n) \), the set of all eigenvectors corresponding to \( \lambda \)

\[
 X^\lambda_1 = \{ x \in \mathbb{R}^n : A x = \lambda x, A \in A_0, x \neq 0 \}
\]

can be described by a system of linear inequalities:

**Theorem 3.** Let \( r > 0 \) and let (i), (ii), (iii) hold. Then for each \( \lambda \in L_1 (i = 1, \ldots, n) \), the set \( X^\lambda_1 \) is given by
\[(A_0 - \lambda E - rr^T r_1) x \leq 0 \]
\[(A_0 - \lambda E + rr^T r_1) x \geq 0 \]
\[x \neq 0.\] (2)

Proof. If \(x \neq 0\), then \(x \in X_1^\lambda\) if and only if \((A - \lambda E)x = 0\) for some \(A - \lambda E \in \{A_0 - \lambda E - rr^T, A_0 - \lambda E + rr^T\}\), which, in turn, is equivalent to \(|(A_0 - \lambda E)x| \leq rr^T|x|\) (Oettli, Prager [1]). Setting \(|x| = T_1x\), we obtain (2). □

In the special case of \(rr^T = \beta e e^T\), \(e = (1, 1, \ldots, 1)^T\), \(\beta > 0\) (uniform tolerances), we have \(D_1 = \beta y_1 y_1^T\) and the normalized eigenvectors from \(X_1^\lambda\) satisfying \(\|x\|_1 = \sum_1 \|x_1\| = 1\) are given simply by

\[-\beta e \leq (A_0 - \lambda E)x \leq \beta e\]
\[y_1^T x = 1.\]

References


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