A Two-Sequence Method for Linear Interval Equations

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Abstract — Zusammenfassung

A Two-Sequence Method for Linear Interval Equations. It is shown that only two matrix sequences are to be constructed to solve a system of linear interval equations with an inverse stable, strongly regular interval matrix.

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Eine Zweisequenzmethode für lineare Gleichungssysteme. Es wird gezeigt, daß nur zwei Matrizensequenzen zu konstruieren sind, um ein lineares Intervallgleichungssystem mit einer invers stabilen, streng regulären Intervallmatrix zu lösen.

It is shown here that some of the author’s previous results can be reformulated in such a way that only two matrix sequences are to be constructed to compute the exact bounds of solution of a system of linear interval equations with an inverse stable, strongly regular interval matrix (see definitions below). The result has two specific features: (i) a componentwise matrix product is used, and (ii) the resulting vectors appear as diagonal vectors of certain matrices.

To introduce the problem in question, assume we are given a linear interval system \( A' x = b' \) with an \( n \times n \) interval matrix \( A' = [A - \Delta, A + \Delta] \), assumed to be regular (\( \det A' \neq 0 \) for each \( A' \in A' \)), and a right-hand side interval vector \( b' = [b - \delta, b + \delta] \).

Wanted is the exact interval solution \( [\underline{x}, \overline{x}] \) where \( \underline{x}, \overline{x} \) are given by

\[
\begin{align*}
\underline{x}_i &= \min \{ x'_i; x' \in M \} \\
\overline{x}_i &= \max \{ x'_i; x' \in M \}, \quad (i = 1, \ldots, n)
\end{align*}
\]

\( M \) being the solution set of \( A' x = b' \).

\[
M = \{ x'; A' x' = b', A' \in A', b' \in b' \}.
\]

An interval matrix \( A' \) is called inverse stable if \( |(A')^{-1}| > 0 \) for each \( A' \in A' \), i.e. if each inverse matrix coefficient preserves its signature over \( A' \). For such an interval matrix, we define a signature matrix \( S \) by
\[ S_{ij} = \begin{cases} 1 & \text{if } A_{ni} > 0 \\ -1 & \text{if } A_{ni} < 0 \end{cases} \quad (i,j=1,\ldots,n) \]

(notice the transposition of indices). For two \( n \times n \) matrices, \( X=(X_{ij}) \) and \( Y=(Y_{ij}) \), we define their componentwise product as \( X \circ Y=(X_{ij}Y_{ij}) \) and the absolute value as \( |X|=\{|X_{ij}\}| \). Further, let \( \text{diag} X \) denote the diagonal vector of \( X \), i.e., \( \text{diag} X=(X_{11},X_{22},\ldots,X_{nn})^T \). Finally, let \( e=(1,1,\ldots,1)^T \in \mathbb{R}^n \) and denote
\[ B=b\ e^T \]
and
\[ D=d\ e^T \]
\((b, d \text{ are column vectors})\). Our method for computing \( x \) and \( \hat{x} \) is based on the following theoretical result.

**Theorem 1:** Let \( A^t \) be regular and inverse stable. Then the matrix equations
\[ AX = B - S \ast (A | X | + D) \]
\[ AX = B + S \ast (A | X | + D) \]
have unique matrix solutions \( \hat{X} \) and \( X \), respectively, and
\[ x = \text{diag} \ X \]
\[ \hat{x} = \text{diag} \ \hat{X} \]

**Proof:** For each \( i \in \{1,\ldots,n\} \), denote by \( y(i) \) the signature vector of the \( i \)-th row of \( A^{-1} \), and let \( T_{y(i)} \) be the diagonal matrix with diagonal vector \( y(i) \) (i.e., \( y(i) = \text{diag} T_{y(i)} \)). In [5], Theorem 1.2, it was proved that under the regularity assumption, the equations
\[ AX - b = - T_{y(i)}(A | x | + d) \]
\[ AX - b = + T_{y(i)}(A | x | + d) \]
have unique (vector) solutions, denoted there by \( x_{y(i)} \) and \( x_{y(i)} \), respectively; moreover, \( x_i = (x_{y(i)},x_{y(i)}) \), \( x_i = (x_{y(i)},x_{y(i)}) \), hold if \( A^t \) is inverse stable ([3], Theorem 3; proved in [4], p. 23). Hence if we define matrices \( X, \hat{X} \) by \( X=(x_{y(1)},\ldots,x_{y(n)}) \) and \( \hat{X}=(x_{y(1)},\ldots,x_{y(n)}) \), then \( X, \hat{X} \) solve uniquely (1), (2) and there holds \( x = \text{diag} \ X, \hat{x} = \text{diag} \ \hat{X} \).

To solve (1) and (2), let us premultiply them first by an approximate inverse \( R \) of the matrix \( A \) to bring them to an equivalent fixed-point form
\[ X = (E - R A) X - R(S \ast (A | X | + D)) + R B \]
\[ \hat{X} = (E - R A) \hat{X} - R(S \ast (A | X | + D)) + R B \]
\((E \text{ is the unit matrix})\). These nonlinear equations may be then solved iteratively by
\[ X^{k+1} = (E - R A) X^k - R(S \ast (A | X^k | + D)) + R B \]
\[ \hat{X}^{k+1} = (E - R A) \hat{X}^k + R(S \ast (A | X^k | + D)) + R B \]
\((k=0,1,\ldots) \) with the recommended starting point
\[ X^0 = \hat{X}^0 = RB. \]
To assure convergence, we shall assume the nonnegative matrix

\[ G = |E - RA| + |R| \]

to satisfy

\[ \rho(G) < 1; \quad (6) \]

this is true if \( A^t \) is strongly regular (i.e., \( \rho(A^{-1}) < 1 \); cf. Neumaier [1]) and \( R \) is a sufficiently close approximation of \( A^{-1} \). We shall later show that (6) guarantees regularity. To be able to check \( A^t \) for inverse stability, we introduce also the matrix

\[ F = (E - G)^{-1} \]

and the condition

\[ GF |R| < |R|. \quad (7) \]

Then the following theorem holds:

**Theorem 2:** Let (6) and (7) hold. Then \( A^t \) is regular and inverse stable and for the sequences \( \{X^k\}, \{\hat{X}^k\} \) generated by (3), (4), (5) we have \( X^k \rightarrow \hat{X}, \hat{X}^k \rightarrow \hat{X} \), with

\[
|X - X^k| \leq GF |X^k - X^{k-1}| \leq G^k F |X^k - X^0| \\
|\hat{X} - \hat{X}^k| \leq GF |\hat{X}^k - \hat{X}^{k-1}| \leq G^k F |\hat{X}^k - \hat{X}^0|
\]

for each \( k \geq 1 \).

**Proof:** First, for each \( A^t \in A^t \) we have

\[ A^t = R^{-1} (E - (E - RA + R(A - A^t))) \]

and since

\[ \rho(E - RA + R(A - A^t)) \leq \rho(G) < 1, \]

we see that \( A^t \) is regular and

\[ ||(A^t)^{-1} - R|| \leq \left( \sum_{i=1}^{\infty} G^i \right) |R| = GF |R| < |R|, \]

which means that \( A^t \) is inverse stable.

Next, arguing as in [2], we first obtain from (4) that

\[ |\hat{X}^{k+1} - \hat{X}^k| \leq G |\hat{X}^k - \hat{X}^{k-1}| \]

for each \( k \geq 1 \) and then by induction

\[ |\hat{X}^{k+m} - \hat{X}^k| \leq G^m + ... + G |\hat{X}^k - \hat{X}^{k-1}| \leq GF |\hat{X}^k - \hat{X}^{k-1}| \leq G^k F |\hat{X}^k - \hat{X}^0| \]

for each \( m \geq 1 \), so that \( \{\hat{X}^k\} \) is a Cauchy sequence, hence \( \hat{X}^k \rightarrow \hat{X} \) due to the uniqueness of the solution of (2). Taking \( m \rightarrow \infty \) in the above inequality, we obtain the estimation for \( |\hat{X} - \hat{X}^k| \). The proof for \( \{X^k\} \) is quite analogous.

Note that the conditions (6) and (7) under which the method works are satisfied if all the coefficients of \( A^{-1} \) are nonzero and \( \delta \) is sufficiently small, which is a frequent case. Also, under these conditions the sequences \( \{X^k\}, \{\hat{X}^k\} \) converge from arbitrary starting points, but the choice (5) seems to be the best since then each column of \( X^0, \hat{X}^0 \) is equal to \( Rb \), an approximate solution to \( Ax = b \).
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