INTERVAL SOLUTIONS OF LINEAR INTERVAL EQUATIONS

Jiri Rohn

(Received June 2, 1989)

Summary. It is shown that if the concept of an interval solution to a system of linear interval equations given by Ratschek and Sauer is slightly modified, then only two nonlinear equations are to be solved to find a modified interval solution or to verify that no such solution exists.

Keywords: linear systems, interval arithmetic.


In this paper we shall deal with the following problem. Given a square interval matrix \( A^\pm = [A^-, A^+] = \{A; A^- \leq A \leq A^+\} \), where \( A^- = (a_{ij}^-) \) and \( A^+ = (a_{ij}^+) \) are \( n \times n \) matrices, and an interval vector \( b^\pm = [b^-, b^+] = \{b; b^- \leq b \leq b^+\} \) with \( b^- = (b_{i}^-) \), \( b^+ = (b_{i}^+) \in \mathbb{R}^n \), find an interval n-vector \( x^\pm = [x^-, x^+] \) such that

\[
\sum_{j=1}^{n} [a_{ij}^-, a_{ij}^+] \cdot [x_j^-, x_j^+] = [b_i^-, b_i^+] \quad (i = 1, \ldots, n)
\]

holds, where the operations involved are performed in interval arithmetic and are generally defined by

\[
[a^-, a^+] + [\beta^-, \beta^+] = [a^- + \beta^-, a^+ + \beta^+]
\]

\[
[a^-, a^+] - [\beta^-, \beta^+] = [a^- - \beta^-, a^+ - \beta^+]
\]

\[
[a^-, a^+] \cdot [\beta^-, \beta^+] = [\min \{a^- \beta^-, a^- \beta^+, a^+ \beta^-, a^+ \beta^+\}, \max \{a^- \beta^-, a^- \beta^+, a^+ \beta^-, a^+ \beta^+\}]
\]

\[
[a^-, a^+] / [\beta^-, \beta^+] = [a^- / \beta^-, a^- / \beta^+], \quad 1 / [\beta^-, \beta^+] = \left[ \frac{1}{\beta^+}, \frac{1}{\beta^-} \right], \quad \text{provided} \ 0 \notin [\beta^-, \beta^+]
\]

where
(for interval arithmetic, see e.g. [4]). This concept of solution was formulated for
interval systems with arbitrary $m \times n$ interval matrices by Ratschek and Sauer [7]
and solved there for the case $m = 1$. It seems that a general solution to (1) is not yet
known; cf. also Nickel [5]. In this paper we shall show that systems of type (1) with
square regular interval matrices can be solved if we impose an additional restriction
upon the concept of a solution in the following sense:

**Definition.** Given $A^t$ (square) and $b^t$, an interval vector $x^t$ is called a strong solution
if it satisfies (1) and if there exist $A', A'' \in A^t$ and $x', x'' \in x^t$ such that $A'x' = b^-$,
$A''x'' = b^+$ hold.

Let us first introduce
\[
A_c = \frac{1}{2}(A^+ + A^-),
\]
\[
\Delta = \frac{1}{2}(A^+ - A^-),
\]
so that $\Delta \geq 0$ and $A^- = A_c - \Delta, A^+ = A_c + \Delta$. We shall show that the problem of
finding a strong solution is closely connected with solving the nonlinear equations
\begin{align}
A_c x - A|x| = b^-,
\end{align}
(2.1)
\begin{align}
A_c x + \Delta|x| = b^+
\end{align}
(2.2)
where $x = (x_j)$ is a real (not interval) vector and the absolute value is defined as
$|x| = (|x_j|)$. We shall need some results about solutions to (2.1), (2.2). A square
interval matrix $A^t$ is called regular if each $A \in A^t$ is nonsingular.

**Theorem 1.** Let $A^t$ be regular. Then the equations (2.1), (2.2) have unique solutions
$x_1$ and $x_2$, respectively.

For the proof of this result, see [8], Theorem 1.2. To solve (2.1) and (2.2), we may
observe that $|x| = Dx$, where $D$ is a diagonal matrix with $D_{ii} = 1$ if $x_i \geq 0$ and
$D_{ii} = -1$ otherwise. Then (2.1) can be written as a system of linear equations
$(A_c - A D) x = b^-$, where $D$ must be found such that $Dx(= |x|) \geq 0$. This is the
underlying idea of the following algorithm:

**Algorithm 1** (for solving (2.1), (2.2)).

**Step 0.** Set $D = E$ (unit matrix).

**Step 1.** Solve $(A_c - A D) x = b^-$ (for (2.2)): $(A_c + A D) x = b^+$.

**Step 2.** If $Dx \geq 0$, set $x_1 := x$ (or, $x_2 := x$) and terminate.

**Step 3.** Otherwise find $k = \min \{j; D_{jj} x_j < 0\}$.

**Step 4.** Set $D_{kk} := -D_{kk}$ and go to **Step 1**.

The algorithm is general, as the following result shows:
Theorem 2. Let $A^t$ be regular. Then Algorithm 1 is finite, passing through Step 1 at most $2^t$ times.

The proof of this theorem can be again found in [8]. Another possibility, though not general, for solving (2.1) (similarly, (2.2)) consists in reformulating (2.1) as a fixed-point equation

$$x = A^{-1}_0 |x| + A^{-1}_0 b^-$$

which may be solved iteratively by

$$x^0 = A^{-1}_0 b^-,$$

$$x^{i+1} = A^{-1}_0 |x^i| + A^{-1}_0 b^- \quad (i = 0, 1, \ldots),$$

but convergence of $\{x^i\}$ to $x_1$ can be established only under the assumption that $\phi(A^{-1}_0 |A|) < 1$, which is not always the case with regular interval matrices; nevertheless, if $A$ is of small norm, then the iterative method is to be preferred.

Returning now back to our original problem of finding a strong solution, we shall show in the next theorem that if strong solutions exist at all, then one of them can be easily expressed by means of the above vectors $x_1, x_2$. Since generally neither $x_1 \leq x_2$, nor $x_1 \geq x_2$ holds, we introduce $\min \{x_1, x_2\}$ as the vector with components $\min \{(x_1)_j, (x_2)_j\}$ $(j = 1, \ldots, n)$, and similarly for $\max \{x_1, x_2\}$.

Theorem 3. Let $A^t$ be regular and let (1) have a strong solution. Then $x^t = [x^-, x^+]$, given by

$$x^- = \min \{x_1, x_2\},$$

$$x^+ = \max \{x_1, x_2\},$$

is also a strong solution.

Proof. Let $x^t$ be a strong solution. Then there exist $A', A'' \in A^t$ and $x', x'' \in x^t$ such that $A'x' = b^-, A''x'' = b^+$ hold. Due to the definition of interval operations, the resulting left-hand side interval vector in (1) contains all vectors of the form $Ax'$, $A \in A^t$. On the other hand, according to the theorem by Oettli and Prager [6], we have $\{Ax'; A \in A^t\} = \{A_0x' - A|x'|, A_0x' + A|x'|\}$. Since $A'x' = b^-$, we conclude that

$$A_0x' - A|x'| = b^-$$

holds, implying $x' = x_1$ in view of the uniqueness of the solution to (2.1) stated in Theorem 1. In a similar way we would obtain that $x'' = x_2$. Now, for $x^t$ given by (3), the interval vector with the components

$$\sum_{j=1}^n [a_{ij}, a_{ij}] \cdot [x_j^-, x_j^+] \quad (i = 1, \ldots, n)$$

is contained in $b^t$ since $x^t \subset x^t$, but also contains $b^-$ and $b^+$ since $x_1, x_2 \in x^t$; hence it equals $b^t$, so that (1) holds and $x^t$ is a strong solution. Q.E.D.
Summing up the results, we can formulate the following algorithm for solving our problem:

**Algorithm 2 (finding a strong solution)**

**Step 1.** Solve (2.1), (2.2) (by Algorithm 1 or iteratively) to find $x_1, x_2$.

**Step 2.** Construct $x'\!$ by (3).

**Step 3.** If $x'$ satisfies (1), stop: $x'$ is a strong solution.

**Step 4.** Otherwise stop: no strong solution exists.

The algorithm works provided $A'\!$ is regular, which is the case e.g. if the spectral radius of $A_A^{-1}$ is less than 1 (Beeck [2]), a condition widely satisfied in practice.

We add two small examples with regular matrices to illustrate the possible outcomes.

**Example 1 (Hansen [3]).** Let

$$A^- = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}, \quad A^+ = \begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix}$$

and $b^- = (0, 60)^T$, $b^+ = (120, 240)^T$. Solving (2.1), (2.2), we obtain

$$x_1 = (0, 30)^T, \quad x_2 = \left(\frac{120}{7}, \frac{480}{7}\right)^T,$$

and

$$x' = \left(\left[0, \frac{120}{7}\right], \left[30, \frac{480}{7}\right]\right)^T$$

satisfies (1), therefore it is a strong solution.

**Example 2 (Barth and Nuding [1]).** Let

$$A^- = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \quad A^+ = \begin{pmatrix} 4 & 1 \\ 2 & 4 \end{pmatrix}$$

and $b^- = (-2, -2)^T$, $b^+ = (2, 2)^T$. Here $x'$ does not satisfy (1), so that no strong solution exists.

A preliminary version of this paper appeared in [9].

**References**


Souhrn

INTERVALOVÁ ŘEŠENÍ SOUSTAV LINEÁRNÍCH INTERVALOVÝCH ROVNIC
Jiří Rohn

Je zavedeno modifikované intervalové řešení soustavy lineárních intervalových rovnic, k jehož výpočtu je třeba vyřešit dvě soustavy nelineárních rovnic.

Резюме

ИНТЕРВАЛЬНЫЕ РЕШЕНИЯ СИСТЕМ ЛИНЕЙНЫХ
ИНТЕРВАЛЬНЫХ УРАВНЕНИЙ

Jiří Rohn

В статье показано, как можно вычислить модифицированное интервальное решение системы линейных интервальных уравнений путём решения двух систем нелинейных уравнений.

Author’s address: RNDr. Jiří Rohn, Matematicko-fyzikální fakulta UK, Malostranské nám. 25, 118 00 Praha 1.