Stability of the optimal basis of a linear program under uncertainty

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Received December 1990
Revised August 1992

We prove that the set of optimal basic variables of a linear program remains stable under mutually independent variations of all data within prescribed tolerances if and only if it is stable for a finite subset of explicitly described linear programs from this family. The cardinality of this subset is exponential in the number of constraints.

linear program; uncertainty; optimal basis; stability

Introduction

In this paper we consider a family of linear programming problems

\[ \text{max}\{c^T x; Ax = b, x \geq 0\} \]  \hspace{1cm} (1)
\[ A \in A', \quad b \in b', \quad c \in c' \]  \hspace{1cm} (2)

with \( A', b', c' \) being given by

\[ A' = \{ A; A_0 - \Delta \leq A \leq A_0 + \Delta\}, \]
\[ b' = \{ b; b_0 - \beta < b < b_0 + \beta\}, \]
\[ c' = \{ c; c_0 - \gamma < c < c_0 + \gamma\}, \]

where \( A_0, \Delta \in \mathbb{R}^{m \times n}, b_0, \beta \in \mathbb{R}^m, c_0, \gamma \in \mathbb{R}^n \) and \( \Delta > 0, \beta > 0, \gamma > 0 \) (componentwise ordering; i.e., \( A' \) is a so-called interval matrix and \( b', c' \) interval vectors) and \( m \leq n \). This means that (1) and (2) describe a family of LP problems of the form (1) with all the data varying independently of each other within some prescribed tolerances (2). This unifying formulation covers two important classes of problems: (a) LP problems under uncertainty, and (b) LP problems with perturbed data.

The problem (1), (2) can be viewed as a parametric optimization problem (\([1,2,4]\)) in which each entry in the data \((A, b, c)\) is parametrized independently. It is the purpose of this paper to study stability of the optimal solution of (1), (2) in the sense of the following definition (see Murty [3] for the linear programming terminology):

**Definition.** Let \( B \) be an \( m \)-tuple of integers from \([1, \ldots, n]\). We call the problem (1), (2) \( B \)-stable if each problem (1) with data satisfying (2) has a nondegenerate basic optimal solution with basic variables \( x_j, j \in B \), and we call it strongly \( B \)-stable if, additionally, each such an optimal solution is unique.

It is obvious that this kind of stability is an important issue from a practical point of view because it implies that the qualitative composition of the optimal solution (e.g. the choice of foods in the famous diet problem) remains constant regardless the actual values of the data within given bounds. In this paper we address the problem from a theoretical point of view. We show in the main theorem that the problem (1), (2) is \( B \)-stable if and only if a finite subset of explicitly given LP problems of the form (1) with data satisfying (2) have the required property. Unfortunately, the cardinality of this subset is \( 2^m \), hence the result is rather of theoretical interest. Even so, the proof is not quite trivial. This author conjectured an existence of some sort
of a finite reduction already in 1979 while writing the paper [5], but was unable to prove it until recently. The impracticality of the result may be overcome by introducing only sufficient stability conditions, but we do not pursue this problem in this paper.

Auxiliary results

In the proof of the main theorem we shall employ three auxiliary results. As two of them are quite recent, we state them here explicitly for convenience of the reader. We introduce the set
\[ Y = \{ y \in \mathbb{R}^n; \ | y_j | = 1 \text{ for } j = 1, \ldots, m \} \]
and for each \( y \in Y \) we denote by \( T_y \) the diagonal matrix with diagonal vector \( y \) (i.e. \( (T_y)_{ii} = y_i \) for each \( i \) and \( (T_y)_{ij} = 0 \) for \( i \neq j \)). Conv \( X \) denotes the convex hull of \( X \).

**Proposition 1** ([7], Theorem 2). Let \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \) and let for each \( y \in Y \) the system of linear inequalities
\[ T_y Ax \geq T_y b \]
have a solution \( x_y \). Then the system of linear equations
\[ Ax = b \]
has a solution in Conv\( (x_y; y \in Y) \).

Next we have two results concerning \( P \)-matrices. Let us recall that a square matrix is called a \( P \)-matrix if all its principal minors are positive [3].

**Proposition 2** ([6], Theorem 1.2). Let \( A^1 \) be a square internal matrix such that each \( A \in A^1 \) is nonsingular. Then \( A_1, A_2 \in A^1 \).

For a given \( p \in (p_i) \in \mathbb{R}^m \) we define \( p^+ = (p_i^+) \) and \( | p | = \max (p_i, 0) \), \( p^- = \max (p_i, 0) \) (i = 1, \ldots, m). Then we have \( p^+ - p^- \leq | p | \leq p^+ + p^- \). The following result is well known (Maruy [3]):

**Proposition 3.** Let \( M \) be a \( P \)-matrix. Then the linear complementarity problem
\[ p^+ = Mp^- + q \]
has a unique solution for each right-hand side vector \( q \).

Main result

Let \( B = (B_1, \ldots, B_m) \) be an \( m \)-tuple of mutually different integers from \( \{1, \ldots, n\} \). As customary in the linear programming theory, for each \( A \in A^1 \) we denote by \( A_B \) the square submatrix of \( A \) consisting of columns with indices \( B_1, \ldots, B_m \), and by \( A_N \) the remaining (nonbasic) part of \( A \). A similar notation also applies to vectors: \( c_B, x_B, c_N, y_B \) etc. In addition to the set \( Y \subset \mathbb{R}^m \) defined above we introduce also the set
\[ Z_B = \{ z \in \mathbb{R}^n; \ | z_j | = 1 \text{ for } j \in B, z_j = 1 \text{ for } j \not\in B, j = 1, \ldots, n \}, \]
so that \( Z_B \) has \( 2^m \) elements. We extend our notation \( T_y \) also to vectors from \( Z_B \) (in this case it denotes an \( n \times n \) matrix). For each \( y \in Y \), \( z \in Z_B \) we define the matrix
\[ A_y = A^0 - T_y \Delta T_y, \]
and the vectors
\[ b_y = b^0 + T_y \beta, \]
\[ c_y = c^0 + T_y \gamma. \]
Notice that for each \( i, j \) we have \((A_{y_{ij}})_{ij} = (A^0 - \Delta)_{ij} \) if \( y_j = 1 \) and \((A_{y_{ij}})_{ij} = (A^0 + \Delta)_{ij} \) otherwise, hence each coefficient of \( A_{y_{ij}} \) is fixed either at the lower, or at the upper bound of the respective entry, and an analogue holds for \( b_y, c_y \). Our characterization of \( R \)-stability is formulated in terms of these quantities:

**Theorem.** The problem (1), (2) is [strongly] \( R \)-stable if and only if for each \( y \in Y \), \( z \in Z_B \) the LP problem
\[ \max \{ c_y x; A_{y_{ij}} x = b_y, x \geq 0 \} \]
has a [unique] nondegenerate basic optimal solution with basic variables \( x_j \), \( j \in B \).

**Comment.** Notice that each LP problem of the form (3) is a 'vertex problem' since its data \((A, b, c)\) form a vertex of the set \( A^1 \times b^1 \times c^1 \) (considered a rectangle in \( \mathbb{R}^{m+n} \)). Not all the 'vertex problems' are included, however, since their number in the worst case \( A > 0, b > 0, c > 0 \) is \( 2^{m+n} \) while there are \( 2^m \) problems of type (3).
**Proof.** The ‘only if’ part is obvious since $A_{xy} \in A^1$, $b \in b^1$ and $c \in c^1$ for each $y \in Y$, $z \in Z_b$. In the proof of the ‘if’ part we shall show that for each $A \in A^1$, $b \in b^1$, $c \in c^1$ there holds: (i) $A_{xy} \neq 0$ is nonsingular, (ii) $A_{xy} x_y = b$ has a positive solution, (iii) the solution of the equation $A_{xy} p = c_y$ satisfies $A_{xy} p \geq c_y$. This will prove that $x$ (with the basic part $x_y$ from (ii) and nonbasic part zero) is a nondegenerate basic optimal solution of (1) with basis index set $B (p$ is then the dual optimal solution). Thus we are confined to prove (i)–(iii).

(i) Let $y \in Y$ and $z = (1, 1, \ldots, 1)' \in Z_b$. Since (3) has a nondegenerate basic optimal solution with basic variables $x_j, j \in B$, we have that the system

$$ (A_{xy}) p - b = 0 $$

has a positive solution $x_y \in \mathbb{R}^m$. Rearranging (4), we see that $x_y$ satisfies

$$ T_y (A_{xy}^0 x_y - b^0) = \Delta_y x_y + \beta $$

Now let $A \in A^1$ and $b \in b^1$. Since $|T_y (A_{xy} - A_{xy}^0) x_y| < \Delta_y x_y$ and $|T_y (b - b^0)| < \beta$, we obtain

$$ T_y (A_{xy}^0 x_y - b^0) = T_y (A_{xy}^0 x_y - b^0) + T_y (A_{xy} - A_{xy}^0) x_y + T_y (b - b^0) $$

$$ \geq \Delta_y x_y + \beta - \Delta_y x_y = \beta = 0 $$

hence

$$ T_y A_{xy} x_y \geq T_y b $$

holds for each $y \in Y$, which in view of Proposition 1 means that $A_{xy} x - b$ has a solution for each $A \in A^1$, $b \in b^1$. Now consider two cases: (a) If $\beta > 0$, then $b^1$ contains $m$ linearly independent vectors, therefore $A_{xy}$ is nonsingular. (b) If $\beta = 0$ for some $i$, take a perturbed vector $b^* = b + (e, e, \ldots, e)'$ where $e > 0$ is chosen so that each of the finitely many systems

$$ (A_{xy}) p - b^* = T_y (e, e, \ldots, e)' $$

preserves a positive solution (this is possible due to nonsingularity of $(A_{xy})$, and nondegeneracy. Since $b^* > 0$, the previous argument applies to prove again that $A_{xy}$ is nonsingular.

(ii) As we have seen in (i), the inequality (5) holds for each $A \in A^1$, $b \in b^1$ and $y \in Y$, hence according to Proposition 1 the unique solution of the equation $A_{xy} x_y = b$ belongs to Conv$(x_y; y \in Y)$ and therefore is positive since each $x_y$ is positive.

(iii) To prove the last assertion, we shall first show that for each $y \in Y$ the nonlinear equation

$$ T_y (A_{xy}^0 p - c_y) = \Delta_y p + \gamma $$

has a unique solution. In fact, using $p = p^* - p^-$ and $\gamma = p^+ + p^-$, (6) can be equivalently rearranged to the form

$$ p^* - p^- = \left[ (A_{xy}^0)^- T_y A_{xy}^- \right]^{-1} \left[ (A_{xy}^0)^+ T_y A_{xy}^+ \right] p^- $$

$$ + \left[ (A_{xy}^0)^- T_y A_{xy}^- \right]^{-1} (c_y + T_y \gamma) $$

which is a linear complementarity problem whose matrix is a P-matrix due to Proposition 2 since both the matrices $(A_{xy}^0)^- T_y A_{xy}^+$ and $(A_{xy}^0)^+ T_y A_{xy}^-$ belong to the interval matrix $(A_{xy}^0; A \in A^1)$ whose elements are all nonsingular according to (b). Hence Proposition 3 gives that (7), and thus also (6), has a unique solution $p_y$. Now, let $A \in A^1$ and $c \in c^1$. We shall first prove that

$$ A_{xy} p_y \geq c_y $$

holds for each $z \in Y$. To this end, for a given $z \in Y$ let a $y \in Y$ as follows: $y_i = 1$ if $(p_y)_i > 0$ and $y_i = 0$ otherwise $(i = 1, \ldots, m)$. Then $|p_y| = T_y p_y$ and substituting into (6) we obtain

$$ (A_{xy}^0 - T_y A_{xy}^-) p_y = c_y + T_y \gamma $$

which can be written as

$$ (A_{xy}^0)^+ p_y = (c_y + T_y \gamma)^+ $$

where the vector $z^+ \in Z_b$ is defined by $z^+_y = z$ and $z^+ = (1, 1, \ldots, 1)$. Then the complementary slackness condition of linear programming gives that $p_y$ is the unique dual optimal solution to the problem (3) for $y, z^+$, hence

$$ A_{xy}^0 p_y \geq (c_y + T_y \gamma)^+ $$

holds and since $\langle (A_{xy} - A_{xy}^0) p_y, c_y \rangle$ we have

$$ A_{xy} p_y = (A_{xy}^0)^+ p_y + (A_{xy} - A_{xy}^0)^+ p_y $$

$$ \geq (A_{xy}^0)^+ p_y - \Delta_y p_y $$

$$ = (A_{xy}^0 - T_y A_{xy})^+ p_y \geq (c_y + T_y \gamma)^+ $$

$$ = c_y + \gamma_y \geq c_y $$

which proves (8). Now, let $p$ be the solution to $A_{xy} p = c_y$ which is unique due to (i). In a similar way as in the part (i) we can prove from (6) that

$$ T_y A_{xy}^0 p_y \geq T_y c_y $
holds for each \( z \in Y \), hence Proposition 1 gives that \( p - \sum_{z \in Y} A_z p_z \) for some real numbers \( \lambda_z \in [0, 1], z \in Y \), satisfying \( \sum_{z \in Y} \lambda_z = 1 \), thus from (8) we finally obtain

\[
A_N p = \sum_{z \in Y} \lambda_z A_z p_z \geq \sum_{z \in Y} \lambda_z c_N = c_N, \tag{10}
\]

which concludes the proof of part (iii).

If, moreover, each problem (3) has a unique optimal solution, then the inequality (9), and consequently also (8) and (10), hold sharply which means that each problem (1) under (2) has a unique nondegenerate basic optimal solution, hence the problem (1), (2) is strongly \( B \)-stable.

\[ \square \]

References