Short communication

A note on solvability of a class of linear complementarity problems

Jiri Rohn

Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic

Received 29 July 1991
Revised manuscript received 21 February 1992

We give a characterization of unique solvability of an infinite family of linear complementarity problems of a special form by means of a finite subset of this family.

Key words: Linear complementarity problem, nonsingular matrix, P-matrix.

A linear complementarity problem is a problem of the form

\[ y = Mz + q \]
\[ y \geq 0, \quad z \geq 0, \]
\[ y^Tz - 0, \]

where \( M \) is an \( n \times n \) matrix and \( q \) an \( n \)-dimensional vector; we shall denote this problem by \( \text{LCP}(M, q) \). A detailed exposition of the linear complementarity theory may be found in Murty’s book [1]. In this note we apply some recent results on systems of linear equations with inexact data [2] to obtain some necessary and sufficient conditions for unique solvability of a whole class of linear complementarity problems of the form \( \text{LCP}(M_1^{-1}M_2, q) \) with \( A \leq M_1 \leq B \) and \( A \leq M_2 \leq B \), where \( A \) and \( B \) are two given \( n \times n \) matrices and \( q \in \mathbb{R}^n \). (Here, as in the sequel, matrix and vector inequalities are understood componentwise and the inverse of a matrix \( M \) is assumed to exist whenever the symbol \( M^{-1} \) is used.)

Before formulating the main result we introduce some notations. A diagonal matrix \( S \) is said to be a signature matrix if each of its diagonal elements is equal to 1 or \(-1\), clearly there are \( 2^n \) signature matrices of size \( n \), among them the unit matrix \( I \). Let \( A, B \) be two \( n \times n \) matrices, \( A \leq B \), and let \( S \) be a signature matrix of the same size. We introduce the matrix

Correspondence to: Jiri Rohn, Faculty of Mathematics and Physics, Charles University, Malostranske nam. 25, 11800 Prague, Czech Republic.
\[ M_S = K_S^{-1} L_S \]

where

\[ K_S = \frac{1}{2}(I + S)A + \frac{1}{2}(I - S)B \]

and

\[ L_S = \frac{1}{2}(I - S)A + \frac{1}{2}(I + S)B. \]

Since \( S \) is a signature matrix, each element of \( K_S \) is equal to the respective element of either \( A \) or \( B \), which implies \( A \leq K_S \leq B \); the same holds for \( L_S \). Further let

\[ q_S = K_S^{-1} Se \]

where \( e = (1, 1, \ldots, 1)^T \). Let us recall that a square matrix is called a \( P \)-matrix if all its principal minors are positive.

Now we have this result:

**Theorem.** Let \( A, B \) be two \( n \times n \) matrices, \( A \leq B \). Then the following assertions are equivalent:

1. Each matrix \( C \) satisfying \( A \leq C \leq B \) is nonsingular.
2. The LCP(\( M_1^{-1}M_2 \), \( q \)) has a unique solution for all matrices \( M_1, M_2 \) satisfying \( A \leq M_1 \leq B \), \( A \leq M_2 \leq B \), and each right-hand side vector \( q \).
3. The LCP(\( M_S \), \( q_S \)) has a solution for each signature matrix \( S \).
4. The system
   \[
   \begin{align*}
   y &= M_S z + q_S, \\
   y \geq 0, \quad z \geq 0,
   \end{align*}
   \]

has a solution for each signature matrix \( S \).

5. \( M \) is a \( P \)-matrix for each signature matrix \( S \).

**Proof.** (i) \( \Rightarrow \) (ii), (i) \( \Rightarrow \) (v): If (i) holds, then according to Theorem 1.2 in [2], each matrix of the form \( M_1^{-1}M_2 \), where \( A \leq M_1 \leq B \) and \( A \leq M_2 \leq B \), is a \( P \)-matrix. This proves (v) due to the definition of \( M_S \) and also implies (ii) in view of the well-known result on unique solvability of a linear complementarity problem LCP(\( M \), \( q \)) with a \( P \)-matrix \( M \), see [1].

(ii) \( \Rightarrow \) (iii) follows from the fact that \( M_S \) is of the form \( M_S = K_S^{-1} L_S \), where \( A \leq K_S \leq B \) and \( A \leq L_S \leq B \).

(iii) \( \Rightarrow \) (iv) is obvious since the solution of LCP(\( M_S \), \( q_S \)) also solves the system (1).

(iv) \( \Rightarrow \) (i): If \( y, z \) solve (1), then they satisfy the system

\[
\begin{align*}
K_S y - L_S z &= Se, \\
y \geq 0, \quad z \geq 0.
\end{align*}
\]
According to the assertion (A2) of Theorem 5.1 in [2], the existence of a solution to a system (2) for an arbitrary signature matrix $S$ implies the nonsingularity of each matrix $C$ satisfying $A \preceq C \preceq B$.

$(v) \Rightarrow (i)$: Follows from the assertion (B1) of Theorem 5.1 in [2]. □

The merit of this result is the fact that unique solvability of an infinite family of linear complementarity problems
\[ \text{LCP}(M^{-1}_1 M_2, q), \]
\[ A \preceq M_1 \preceq B, \]
\[ A \preceq M_2 \preceq B, \]
\[ q \in \mathbb{R}^n, \]

can be characterized by means of a finite subset of this family (equivalence (ii)$\iff$(iii)). But even more, as the assertion (iv) shows, the existence of nonnegative solutions to a finite number of systems of linear equations of the type (1) (where the complementarity constraint is dropped) is sufficient for unique solvability of each problem in the family (3); however, the number of test problems is exponential in matrix size. Nevertheless, there exists a verifiable sufficient condition: if
\[ \rho \left( |2I - Q(A + B)| + |Q|(B - A) \right) < 2 \]
holds for some (but arbitrary) $n \times n$ matrix $Q$ (where $\rho$ is the spectral radius and $| \cdot |$ denotes the absolute value of a matrix), then each matrix $C$ satisfying $A \preceq C \preceq B$ is nonsingular [3], hence each problem in the family (3) is uniquely solvable. As explained in [3], for practical verification it is recommended to choose $Q$ as the computed value of $(\frac{1}{2}(A + B))^{-1}$. Notice also that if (3) contains a problem which is not uniquely solvable, then there exists a signature matrix $S$ such that either $K_S$ is singular, or $\text{LCP}(M_S, q_S)$ does not possess a solution (assertion (iii)).

Linear complementarity problems of the form $\text{LCP}(M^{-1}_1 M_2, q), A \preceq M_1 \preceq B, A \preceq M_2 \preceq B$ arise naturally in solving systems of linear equations with inexact data; see [2] for details.

References

