Checking Bounds on Solutions of Linear Interval Equations is NP-Hard

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Dedicated to Prof. M. Fiedler and Prof. V. Pták on the occasion of their retirement

Submitted by

ABSTRACT

We prove that it is NP-hard to decide whether the solution set of a system of linear interval equations is contained in a given interval vector, even in the case that the system matrix is strongly regular.

1. INTRODUCTION

Consider a system of linear interval equations

\[ A'Ix = b'I \quad (1.1) \]

where

\[ A'I = [A_c - \Delta, A_c + \Delta] := \{A'; A_c - \Delta \leq A' \leq A_c + \Delta\} \]

is an interval matrix and

\[ b'I = [b_c - \delta, b_c + \delta] := \{b'; b_c - \delta \leq b' \leq b_c + \delta\} \]

*Work supported by Charles University Grant GAUK 357
is an interval vector; here, $A_c, \Delta \in R^{n \times n}$, $b_c, \delta \in R^n$ and $\Delta \geq 0, \delta \geq 0$ (matrix and vector inequalities are understood componentwise). The solution set of the system (1.1) is defined by 

$$X(A^I, b^I) = \{ x; A'x = b' \text{ for some } A' \in A^I, b' \in b^I \}.$$ 

It is well known (cf. e.g. Neumaier [4]) that the solution set is of a complicated nonconvex structure in general. Therefore the problem of solving (1.1) is usually formulated as follows (see [1], [4]): find an interval vector $x^I$ (as narrow as possible) satisfying

$$X(A^I, b^I) \subset x^I$$ (1.2)

(provided $X(A^I, b^I)$ is bounded), or verify that $X(A^I, b^I)$ is unbounded.

It has been proved recently that this problem is NP-hard [8]. Let us recall that a problem is called NP-hard if each problem in the class NP can be polynomially reduced to it. Thus, unless the class NP of problems solvable by nondeterministic polynomial-time algorithms is equal to the class P of problems solvable by polynomial-time algorithms, which is currently considered highly unlikely (see Garey and Johnson [3] for details), there does not exist a polynomial-time algorithm for solving an NP-hard problem.

In this paper we address another related problem: for a system (1.1), given an $n$-dimensional interval vector $x^I$, check whether (1.2) holds, or not. We shall prove that this problem is NP-hard even for a very restricted class of systems with strongly regular interval matrices. Let us recall that an interval matrix $A^I = [A_c - \Delta, A_c + \Delta]$ is called strongly regular [4] if $A_c$ is nonsingular and

$$\rho(|A_c^{-1}|\Delta) < 1$$ (1.3)

holds (here, $\rho$ denotes the spectral radius and for $A = (a_{ij})$, $|A|$ is defined by $|A| = (|a_{ij}|)$). A well-known result by Beeck [2] states that if $A^I$ satisfies (1.3), then $A^I$ is regular (i.e., each $A \in A^I$ is nonsingular). The problems (1.1) with strongly regular matrices are usually considered “tractable”, for several reasons: 1) regularity of $A^I$ can be easily checked (whereas it is an NP-hard problem in general [5]), 2) a vector $x^I$ satisfying (1.2) can be found in polynomial time [4], [7] (whereas the enclosure problem is again NP-hard in general [8]), and 3) several iterative methods (as Rump [10], Rohn [6]) are guaranteed to converge under (1.3) (and convergence is generally not preserved if (1.3) is violated). Nevertheless, the main result of this paper shows that checking bounds on solutions is difficult even in this special case:

**Theorem 1.1.** The following decision problem is NP-hard:
Instance. A strongly regular interval matrix $A^I$ and interval vectors $b^I, x^I$ (with rational bounds).

Question. Is $X(A^I, b^I) \subset x^I$?

The proof of this theorem will be carried out in section 3, employing two auxiliary results established in section 2. The recent NP-hardness result for computing the optimal (i.e., narrowest) bounds on $X(A^I, b^I)$ (Rohn and Kreinovich [9]) then becomes an easy consequence of Theorem 1.1.

2. AUXILIARY RESULTS

In this section we describe an auxiliary construction and prove its properties to be used later in the proof of the main theorem.

Let $A$ be an arbitrary real nonsingular $n \times n$ matrix and $\beta$ a positive real number. We shall consider an $(n+1) \times (n+1)$ interval matrix introduced in [9] and defined as follows:

$$A^I_\beta = [A_c - \Delta, A_c + \Delta]$$  \hspace{1cm} (2.1)

with

$$A_c = \begin{pmatrix} A^{-1} & 0 \\ (A^{-1})_n & -1 \end{pmatrix}$$  \hspace{1cm} (2.2)

and

$$\Delta = \begin{pmatrix} \beta e e^T & 0 \\ 0^T & 0 \end{pmatrix}$$  \hspace{1cm} (2.3)

where $(A^{-1})_n$ denotes the $n$-th row of $A^{-1}$ and $e = (1,1,\ldots,1)^T \in \mathbb{R}^n$. Let us additionally denote by $I$ the unit matrix and by $e_n$ the vector $(0,0,\ldots,0,1)^T \in \mathbb{R}^n$.

Proposition 2.1. Let $A$ be nonsingular and let $\beta$ satisfy

$$0 < \beta < \frac{1}{e^T|A|e}.$$  \hspace{1cm} (2.4)

Then the interval matrix $A^I_\beta$ given by (2.1)-(2.3) is strongly regular and each $A' \in A^I_\beta$ satisfies

$$|(A')^{-1} - A_c^{-1}| \leq \frac{\beta}{1 - \beta e^T|A|e} \begin{pmatrix} |A|e e^T|A| & 0 \\ e^T|A| & 0 \end{pmatrix}. \hspace{1cm} (2.5)$$
Proof. First, a direct computation shows that

\[
\begin{pmatrix}
A^{-1} & 0 \\
(A^{-1})_{n} & -1
\end{pmatrix}
\begin{pmatrix}
A & 0 \\
e_{n}^{T} & -1
\end{pmatrix}
= 
\begin{pmatrix}
I & 0 \\
0^{T} & 1
\end{pmatrix},
\]

hence

\[
A_{e}^{-1} = \begin{pmatrix} A & 0 \\ e_{n}^{T} & -1 \end{pmatrix},
\]

so that for \( D = |A_{e}^{-1}| \Delta \) we have

\[
D = \beta \begin{pmatrix} [A]ee^{T} & 0 \\ e^{T} & 0 \end{pmatrix}.
\]

Let \( Dx' = \lambda x' \) for some (complex) \( \lambda \) and \( x' \neq 0 \). Let \( x' = (x^{T}, x_{n+1})^{T} \), then we have

\[
\beta [A]ee^{T} x = \lambda x, \quad (2.6)
\]

\[
\beta e^{T} x = \lambda x_{n+1}.
\]

Hence, if \( e^{T} x = 0 \), then \( \lambda = 0 \); if \( e^{T} x \neq 0 \), then from (2.6) we obtain \( \lambda = \beta e^{T} [A]e \). Thus \( \rho(D) < 1 \) due to (2.4), so that \( A_{\beta}' \) is strongly regular.

Next, a simple computation gives that \( D^{2} = \beta (e^{T} [A]e)D \), hence \( D^{j} = \beta^{j-1} (e^{T} [A]e)^{j-1}D \) for \( j \geq 1 \). Then for each \( A' \in A_{\beta}' \), from the identity \( A' = A_{e} (I - A_{e}^{-1} (A_{e} - A')) \), in view of the fact that \( \rho(A_{e}^{-1} (A_{e} - A')) \leq \rho(D) < 1 \) we obtain

\[
|A' - A_{e}^{-1}| \leq \sum_{1}^{\infty} D^{j} |A_{e}^{-1}| = \sum_{1}^{\infty} \beta^{j-1} (e^{T} [A]e)^{j-1} D |A_{e}^{-1}|
\]

\[
= \frac{\beta}{1 - \beta e^{T} [A]e} \begin{pmatrix} [A]ee^{T} [A] & 0 \\ e^{T} [A] & 0 \end{pmatrix},
\]

which proves (2.5). \( \square \)

In the next proposition we shall consider the solution set of the system

\[
A_{\beta}'x = b_{\beta}'
\]

(2.7)

where \( A_{\beta}' \) is as above and \( b_{\beta}' \) is the \((n + 1)\)-dimensional interval vector

\[
b_{\beta}' = \left[ \begin{pmatrix} -\beta e \\ 0 \end{pmatrix}, \begin{pmatrix} \beta e \\ 0 \end{pmatrix} \right].
\]

(2.8)

Denote \( e' = (e^{T}, 1)^{T} \in R^{n+1} \).
Proposition 2.2. Let $A$ be nonsingular, $L$ a nonnegative integer, and let $\beta$ satisfy
\[ 0 < \beta < \frac{1}{\max\{e^T|A|e, L\}}. \] (2.9)

Then
\[ z^T Ay > L \] (2.10)
holds for some $z,y \in \{-1,1\}^n$ if and only if the solution set $X(A_I^l, b_I^l)$ of the system (2.7) does not satisfy
\[ X(A_I^l, b_I^l) \subset x', \] (2.11)
where $x' = [\tilde{x}, \tilde{x}]$ is given by
\[ \tilde{x} = -\nu e' \] (2.12)
\[ \tilde{x} = \left( \nu e^T, \frac{\beta}{1 - \beta e^T|A|e} \right)^T \] (2.13)
and
\[ \nu = \frac{\beta \max\{\|A\|_{\infty}, 1\}}{1 - \beta e^T|A|e}. \] (2.14)

Proof. Let $x \in X(A_I^l, b_I^l)$, so that $A'x = b'$ for some $A' \in A_I^l, b' \in b_I^l$. Then from Proposition 1 we have

\[
|z| \leq |A_c^{-1}| \cdot |b'| + |(A')^{-1} - A_c^{-1}| \cdot |b'|
\leq \left( \begin{array}{c} |A| \\ e^T \\ 1 \end{array} \right) \left( \begin{array}{c} 0 \\ \beta e \\ 0 \end{array} \right) + \frac{\beta}{1 - \beta e^T|A|e} \left( \begin{array}{c} |A|e^T|A| \\ e^T|A| \\ 0 \end{array} \right) \left( \begin{array}{c} 0 \\ \beta e \end{array} \right)
\leq \frac{\beta}{1 - \beta e^T|A|e} \left( \begin{array}{c} |A|e \\ 0 \end{array} \right) \leq \nu e',
\]

hence $X(A_I^l, b_I^l) \subset [-\nu e', \nu e']$. By comparing this with (2.12) and (2.13), we see that $X(A_I^l, b_I^l) \subset [\tilde{x}, \tilde{x}]$ does not hold if and only if

\[ \mu := \max\{x_{n+1}'; x' \in X(A_I^l, b_I^l)\} > \frac{\beta}{1 - \beta L}. \]

Let $x' \in R^{n+1}$. Then from the construction of $A_c$ and $\Delta$ it follows that $x' \in X(A_I^l, b_I^l)$ if and only if it is of the form $x' = (x^T, x_{n+1})^T$, where
\[ x \in X([A^{-1} - \beta ee^T, A^{-1} + \beta ee^T], [-\beta e, \beta e]) \text{ and } x_{n+1} = (A^{-1})_n x. \]

Then from Proposition 2 in [9] we have

\[ \mu = \frac{\beta}{1 - \beta \max\{z^T Ay; z, y \in \{-1,1\}^n\}}, \]

hence \( \mu > \frac{\beta}{1 - \beta} \) if and only if \( z^T Ay > L \) for some \( z, y \in \{-1,1\}^n \). Hence, (2.10) is true if and only if (2.11) does not hold.

3. PROOF OF THE MAIN RESULT

We shall now prove the main result formulated in section 1 as a consequence of Proposition 2.

Proof of Theorem 1.1. In [5], Thm. 2.6 it is proved that the decision problem

Instance. A nonsingular rational matrix \( A \) and a nonnegative integer \( L \).

Question. Is \( z^T Ay > L \) for some \( z, y \in \{-1,1\}^n \)?

is NP-hard. We shall polynomially reduce this problem to that one formulated in Theorem 1.1. Given a nonsingular rational matrix \( A \) and a nonnegative integer \( L \), choose a rational number \( \beta \) satisfying (2.9), invert \( A \) and construct \( A_I^\beta, b_I^\beta \) and \( x^I \) by (2.1)-(2.3), (2.8) and (2.12)-(2.14); this can be done in polynomial time, and \( A_I^\beta \) is strongly regular (Proposition 1).

Now, if (2.11) is true, then the solution to the above decision problem is “no”, and if (2.11) is false, then the solution is “yes” (Proposition 2). In this way we have polynomially reduced the above-formulated NP-hard problem to that one of Theorem 1.1, hence the latter problem is NP-hard as well.

In [9] it is proved that computing the narrowest interval vector \( x^I_{opt} \) containing the solution set \( X(A^I, b^I) \) is NP-hard. This can be now proved as a simple consequence of Theorem 1.1. The narrowest interval vector \( x^I_{opt} = [x, x] \) is obviously given by

\[
\begin{align*}
x_i & = \min \{ x_i; x \in X(A^I, b^I) \}, \\
x_i & = \max \{ x_i; x \in X(A^I, b^I) \}
\end{align*}
\]

(\( i = 1, \ldots, n \)). Thus for an arbitrary \( x^I \),

\[ X(A^I, b^I) \subset x^I \]

is true if and only if

\[ x^I_{opt} \subset x^I \]

holds. Hence, the decision problem of Theorem 1.1 can be polynomially reduced to that of computing \( x^I_{opt} \), which is then NP-hard.
REFERENCES


7 J. Rohn, Cheap and tight bounds: the recent result by E. Hansen can be made more efficient, to appear in *Interval Computations*.

8 J. Rohn, Enclosing solutions of linear interval equations is NP-hard, to appear in *Computing*.
