INTERVAL $P$–MATRICES

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Abstract. A characterization of interval $P$–matrices is given. The result implies that a symmetric interval matrix is a $P$–matrix if and only if it is positive definite (although nonsymmetric matrices may be involved). As a consequence it is proved that the problem of checking whether a symmetric interval matrix is a $P$–matrix is NP–hard.

Key words. interval matrix, $P$–matrix, positive definiteness, NP–hardness

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1. Introduction. As is well known, an $n \times n$ matrix $A$ is called a $P$–matrix if all its principal minors are positive. $P$–matrices play an important role in several areas, e.g. in the linear complementarity theory since they guarantee existence and uniqueness of the solution of a linear complementarity problem (see Murty [6]).

A basic characterization of $P$–matrices was given by Fiedler and Pták [3]: $A$ is a $P$–matrix if and only if for each $x \in \mathbb{R}^n, x \neq 0$ there exists an $i$ such that $x_i(Ax)_i > 0$ holds. This result immediately implies that a symmetric matrix $A$ is a $P$–matrix if and only if it is positive definite (Wilkinson [13]). In fact, if $A$ is positive definite, then for each $x \neq 0$, from $\sum_i x_i(Ax)_i = x^T Ax > 0$ it follows that $x_i(Ax)_i > 0$ for some $i$, hence $A$ is a $P$–matrix; conversely, if $A$ is a $P$–matrix, then all its leading principal minors are positive, hence it is positive definite in view of the Sylvester determinant criterion [6].

In this paper we focus our attention on interval $P$–matrices. An interval matrix $A^I = [A, \overline{A}] = \{A; A \leq A \leq \overline{A}\}$, where $A$ and $\overline{A}$ are $n \times n$ matrices satisfying $A \leq \overline{A}$ (componentwise), is said to be a $P$–matrix if each $A \in A^I$ is a $P$–matrix. In section 2 we introduce a finite set of matrices $A_z$ in $A^I$ (whose cardinality is at most $2^{n-1}$) such that $A^I$ is a $P$–matrix if and only if all the matrices $A_z$ are $P$–matrices (Theorem 2.3)). In view of a similar characterization of positive definiteness of $A^I$ via the matrices $A_z$ (Theorem 2.4), it is then proved in section 3 that a symmetric interval matrix $A^I$ (i.e., with symmetric bounds $A, \overline{A}$) is a $P$–matrix if and only if it is positive definite (Theorem 3.2). This is a generalization of the above result for real symmetric matrices, but it is not a simple consequence of it since here nonsymmetric matrices may be involved. As a consequence of this result we obtain that the problem of checking whether a symmetric interval matrix is a $P$–matrix is NP–hard (Theorem 3.4). This result shows that the exponential number of test matrices $A_z$ used in the necessary and sufficient condition of Theorem 2.3 is highly unlikely to be essentially reducible.

2. Characterizations. Let us introduce an auxiliary set

$$Z = \{z \in \mathbb{R}^n; z_j \in \{-1, 1\} \text{ for } j = 1, \ldots, n\},$$

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i.e. the set of all $\pm 1$-vectors. The cardinality of $Z$ is obviously $2^n$. For an interval matrix

$$A^I = [A, A],$$

we define matrices $A_z, z \in Z$ by

$$(A_z)_{ij} = \frac{1}{2}(A_{ij} + A_{ij}) - \frac{1}{2}(A_{ij} - A_{ij})z_iz_j$$

($i, j = 1, \ldots, n$). Clearly, $(A_z)_{ij} = A_{ij}$ if $z_iz_j = 1$ and $(A_z)_{ij} = A_{ij}$ if $z_iz_j = -1$, hence $A_z \in A^I$ for each $z \in Z$, and the number of mutually different matrices $A_z$ is at most $2^{n-1}$ (since $A_{-z} = A_z$ for each $z \in Z$), and equal to $2^{n-1}$ if $A < A$. The properties in question ($P$-property and positive definiteness) will be formulated below in terms of the finite set of matrices $A_z, z \in Z$. For a vector $x \in R^n$, let us define its sign vector

$$z = \text{sgn} x$$

by

$$z_i = \begin{cases} 1 & \text{if } x_i \geq 0 \\ -1 & \text{if } x_i < 0 \end{cases}$$

($i = 1, \ldots, n$), so that $\text{sgn} x \in Z$. For a matrix $A = (A_{ij})$ we introduce its absolute value by $|A| = (|A_{ij}|)$; a similar notation also applies to vectors.

The basic property of the matrices $A_z, z \in Z$, is summed up in the following auxiliary result; notice that no assumptions on $A^I$ are made.

**Theorem 2.1.** Let $A^I$ be an $n \times n$ interval matrix, $x \in R^n$, and let $z = \text{sgn} x$. Then for each $A \in A^I$ and each $i \in \{1, \ldots, n\}$ we have

$$(1) \quad x_i(Ax)_i \geq x_i(A_zx)_i.$$

**Proof.** Let $A \in A^I$ and $i \in \{1, \ldots, n\}$. Then

$$|x_i(Ax)_i - x_i((\frac{1}{2}(A + A)x)_i| = |x_i((A - \frac{1}{2}(A + A))x)_i|$$

$$\leq |x_i|(|A - \frac{1}{2}(A + A)||x|| \leq |x_i|(\frac{1}{2}(A - A)|x||,

hence

$$x_i(Ax)_i \geq x_i((\frac{1}{2}(A + A)x)_i - |x_i|(\frac{1}{2}(A - A)|x||.$$

Since $z = \text{sgn} x$, we have $|x_j| = z_jx_j$ for each $j$, hence

$$x_i(Ax)_i \geq \sum_j (\frac{1}{2}(A_{ij} + A_{ij}) - \frac{1}{2}(A_{ij} - A_{ij})z_iz_j)x_ix_j$$

$$= \sum_j (A_z)_{ij}x_ix_j = x_i(A_zx)_i.$$
which concludes the proof. \qed

As the first consequence of this result, we prove a Fiedler–Ptáček type characterization of interval \(P\)-matrices. Notice that the inequality holds “uniformly” here:

**Theorem 2.2.** An interval matrix \(A^I\) is a \(P\)-matrix if and only if for each \(x \in \mathbb{R}^n, x \neq 0\), there exists an \(i \in \{1, \ldots, n\}\) such that

\[
x_i(Ax)_i > 0
\]

holds for each \(A \in A^I\).

**Proof.** If (2) holds, then each \(A \in A^I\) is a \(P\)-matrix by the Fiedler-Ptáček theorem. Conversely, let \(A^I\) be a \(P\)-matrix and let \(x \neq 0\). Put \(z = \text{sgn} \, x\), then \(A_z\) is a \(P\)-matrix, hence by the Fiedler-Ptáček theorem we have \(x_i(A_zx)_i > 0\) for some \(i\). Then (1) implies \(x_i(Ax)_i \geq x_i(A_zx)_i > 0\) for each \(A \in A^I\), and we are done. \(\Box\)

The following characterization, however, turns out to be much more useful:

**Theorem 2.3.** \(A^I\) is a \(P\)-matrix if and only if each \(A_z, z \in Z\), is a \(P\)-matrix.

**Proof.** If \(A^I\) is a \(P\)-matrix, then each \(A_z \in A^I\) is obviously also a \(P\)-matrix. Conversely, let each \(A_z, z \in Z\), be a \(P\)-matrix. Let \(x \in \mathbb{R}^n, x \neq 0\), and let \(z = \text{sgn} \, x\). Since \(A_z\) is a \(P\)-matrix, there exists an \(i\) with \(x_i(A_zx)_i > 0\), then from Theorem 2.1 we obtain \(x_i(Ax)_i \geq x_i(A_zx)_i > 0\) for each \(A \in A^I\), hence \(A^I\) is a \(P\)-matrix by Theorem 2.2. \(\Box\)

Another finite characterization of interval \(P\)-matrices, formulated in different terms, was proved by Bialas and Garloff [1].

In analogy with the terminology introduced for \(P\)-matrices, an interval matrix \(A^I\) is said to be positive definite if each \(A \in A^I\) is positive definite (i.e., satisfies \(x^T Ax > 0\) for each \(x \neq 0\)). The following theorem was proved in [9, Thm. 2]. We give here another proof of this result to make the paper self-contained and to demonstrate that it is a simple consequence of Theorem 2.1:

**Theorem 2.4.** \(A^I\) is positive definite if and only if each \(A_z, z \in Z\), is positive definite.

**Proof.** The “only if” part is obvious since \(A_z \in A^I\) for each \(z \in Z\). To prove the “if” part, take an \(A \in A^I\) and \(x \in \mathbb{R}^n, x \neq 0\). For \(z = \text{sgn} \, x\), from Theorem 2.1 we have

\[x_i(Ax)_i \geq x_i(A_zx)_i\]

for each \(i\), hence

\[x^T Ax = \sum_i x_i(Ax)_i \geq \sum_i x_i(A_zx)_i = x^T A_zx > 0,
\]

so that \(A\) is positive definite. Thus, by definition, \(A^I\) is positive definite. \(\Box\)

The last two theorems reveal that both the \(P\)-property and positive definiteness of interval matrices are characterized by the same finite subset of matrices \(A_z \in A^I, z \in Z\). This relationship will become even more apparent in the case of symmetric interval matrices which we shall consider in the next section.

**3. Symmetric interval matrices.** For an interval matrix \(A^I = [\underline{A}, \overline{A}]\), define an associated interval matrix \(A^I_s\) by

\[A^I_s = \left[ \frac{1}{2}(\underline{A} + \underline{A}^T), \frac{1}{2}(\overline{A} + \overline{A}^T) \right].\]
$A^T$ is called symmetric if $A^T = A^T_1$, which is clearly the case if and only if both $A$ and $A$ are symmetric. Hence, $A^T_1$ is always a symmetric interval matrix. The relationship between positive definiteness and $P$–property is provided by the following theorem:

**Theorem 3.1.** $A^T$ is positive definite if and only if $A^T_1$ is a $P$–matrix.

**Proof.** For each $z \in Z$, let us denote by $A^z$ the matrix $A_z$ for $A^T_1$, i.e.

$$
(A^z)_{ij} = \frac{1}{4}(A_{ij} + A_{ji} + \overline{A}_{ij} + \overline{A}_{ji}) - \frac{1}{4}(\overline{A}_{ij} + A_{ji} - A_{ij} - A_{ji})z_iz_j
$$

$(i, j = 1, \ldots, n)$. Then $A^z$ is symmetric and a direct computation shows that

$$
x^TA^zx = x^TAzx
$$

holds for each $x \in R^n$. Now, if $A^T$ is positive definite, then each $A_z, z \in Z$ is positive definite, hence each $A^z$ is positive definite due to (3), so that $A^z$ is a $P$–matrix, hence $A^T_1$ is a $P$–matrix by Theorem 2.3. Conversely, if $A^T_1$ is a $P$–matrix, then each $A^z, z \in Z$ is a $P$–matrix, hence it is positive definite due to its symmetry, thus each $A_z, z \in Z$ is positive definite by (3) and $A^T$ is positive definite by Theorem 2.4.

Our main result on symmetric interval matrices is now obtained as a simple consequence of Theorem 3.1.

**Theorem 3.2.** A symmetric interval matrix $A^T$ is a $P$–matrix if and only if it is positive definite.

**Proof.** The result follows immediately from Theorem 3.1 since a symmetric interval matrix $A^T$ satisfies $A^T = A^T_1$ by definition.

At the beginning of the Introduction we showed that a real symmetric matrix is a $P$–matrix if and only if it is positive definite. The result of Theorem 3.2 sounds verbally alike, but it is not a simple consequence of the real case since here nonsymmetric matrices may be involved. In fact, it can be immediately seen that a symmetric interval matrix $A^T = [A, \overline{A}]$ contains nonsymmetric matrices if and only if $A_{ij} < \overline{A}_{ij}$ holds for some $i \neq j$.

An interval matrix $A^T$ is called regular (cf. Neumaier [7]) if each $A \in A^T$ is nonsingular. The following result shows that for symmetric interval matrices the $P$–property is preserved by regularity. Several other results of this type are summed up in [10].

**Theorem 3.3.** A symmetric interval matrix $A^T$ is a $P$–matrix if and only if it is regular and contains at least one symmetric $P$–matrix.

**Proof.** A symmetric interval $P$–matrix $A^T$ is regular (each $A \in A^T$ has a positive determinant) and contains a symmetric $P$–matrix $A$. If $A^T$ is regular and contains a symmetric $P$–matrix $A_0$, then $A_0$ is positive definite, hence $A^T$ is positive definite by Theorem 3 in [9], which in the light of Theorem 3.2 means that $A^T$ is a $P$–matrix.

Another relationship between regularity and $P$–property of interval matrices was established in [8, Thm. 5.1, assert. (B1)]: an interval matrix $A^T = [A, \overline{A}]$ is regular if and only if $(A + \overline{A} - S(\overline{A} - A))^{-1}(A + \overline{A} + S(\overline{A} - A))$ is a $P$–matrix for each signature matrix $S$ (i.e., a diagonal matrix with $\pm 1$ diagonal elements). This topic was recently studied by Johnson and Tsatsomeros [5].

The necessary and sufficient condition of Theorem 2.3 employs up to $2^{n-1}$ test matrices $A_z, z \in Z$. There is a natural question whether an essentially simpler criterion can be found. The following Theorem 3.4 gives an indirect answer to this question: it implies that an existence of a polynomial-time algorithm for checking the $P$–property of symmetric interval matrices would imply that the complexity classes P and NP are equal, thereby running contrary to the current (unproved) conjecture.
that $P \neq NP$. We refer the reader to the classical book by Garey and Johnson [4] for a detailed discussion of the problem “$P=NP$” and related issues.

**Theorem 3.4.** The following problem is NP-hard:

*Instance.* A symmetric interval matrix $A^I = [\underline{A}, \overline{A}]$ with rational bounds $\underline{A}, \overline{A}$.

*Question.* Is $A^I$ a $P$–matrix?

*Proof.* By Theorem 3.2, $A^I$ is a $P$–matrix if and only if it is positive definite; checking positive definiteness of symmetric interval matrices was proved to be NP-hard in [11].

Coxson [2] proved that the $P$–matrix problem for real matrices is co-NP-complete. His result concerns nonsymmetric matrices since the symmetric case can be solved by Sylvester determinant criterion which can be performed in polynomial time (Schrijver [12]). Theorem 3.4 shows that for interval matrices even the symmetric case is NP-hard.

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