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Complexity of Solving Linear Interval Equations

It is proved that computing enclosures of solutions of linear interval equations with overestimation bounded by a polynomial in the system size is NP-hard.

1. Introduction

Solving linear interval equations usually means computing enclosures. For a system of linear interval equations

\[ A^l x = b^l \] (\(A^l\) square), enclosure is defined as an interval vector \([\underline{y}, \overline{y}]\) satisfying

\[ X \subseteq [\underline{y}, \overline{y}] , \]

where \(X\) is the solution set:

\[ X = \{x; \ Ax = b\ \text{for some } A \in A^l, \ b \in b^l\} . \]

Various enclosure methods can be found in Alefeld and Herzberger [1]. If \(A^l\) is regular (i.e., each \(A \in A^l\) is nonsingular), then there exists the narrowest enclosure \([\underline{x}, \overline{x}]\) given by

\[ \underline{x}_i = \min_{X} x_i, \]

\[ \overline{x}_i = \max_{X} x_i \]

for each \(i\). Computing \([\underline{x}, \overline{x}]\) was proved to be NP-hard (Rohn and Kreinovich [7]; also, Kreinovich, Lakeyev and Noskov [4] for the rectangular case). In the main result of this paper we show that computing enclosures with overestimation bounded by a polynomial in the system size is NP-hard. The result holds true even for a very restricted class of systems (1) with \(A^l = [A_c - \Delta, A_c + \Delta]\) having nondegenerate interval coefficients in one row only and satisfying \(\rho(A_c^{-1}|\Delta) = 0\). Hence, the problem of computing sufficiently narrow enclosures turns out to be more difficult than previously believed.

2. Preliminaries

A real symmetric \(n \times n\) matrix \(A = (a_{ij})\) is called an MC-matrix [5] if it is of the form

\[ a_{ij} \begin{cases} = n & \text{if } i = j \\ \in \{0, -1\} & \text{if } i \neq j \end{cases} \]

\((i, j = 1, \ldots, n)\). In the proof of the main theorem we shall essentially utilize the following result ([6], Corollary 7) concerning the norm

\[ \|A\|_{\infty, 1} = \max\{\|Ax\|_1; \ \|x\|_{\infty} = 1\} \]

(where \(\|x\|_1 = \sum_i |x_i|\) and \(\|x\|_{\infty} = \max_i |x_i|\); see Golub and van Loan [3], p. 15):

**Proposition 1.** Computing \(\|A\|_{\infty, 1}\) is NP-hard for MC-matrices.

Next we introduce a class of systems (1) of a special form. For each pair of rational numbers \(\varepsilon > 0, \ \delta > 0\) we shall denote by \(H_{\varepsilon, \delta}\) the family of systems of linear interval equations

\[ A^l x = b^l \]
with $A^I$ of the form

$$A^I = \begin{pmatrix} a & -\varepsilon e^T, \varepsilon e^T \\ 0 & A^{-1} \end{pmatrix},$$

where $a$ is a positive rational number, $A$ is an $n \times n$ MC-matrix ($n$ arbitrary, $n \geq 1$), $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$ (i.e., $A^I$ is $(n + 1) \times (n + 1)$), and

$$b^I = \begin{pmatrix} 0 \\ [-\delta e, \delta e] \end{pmatrix}$$

is an $(n + 1)$-dimensional interval vector. If we write (2) as

$$A^I = [A_c - \Delta, A_c + \Delta],$$

then

$$A_c = \begin{pmatrix} a & 0 \\ 0 & A^{-1} \end{pmatrix}$$

is nonnegative symmetric positive definite [5], the radius matrix

$$\Delta = \begin{pmatrix} 0 & \varepsilon e^T \\ 0 & 0 \end{pmatrix}$$

has nonzero coefficients in the first row only, and

$$|A_c^{-1}|\Delta = \begin{pmatrix} 0 & \frac{\varepsilon}{\pi} e^T \\ 0 & 0 \end{pmatrix},$$

hence

$$\varrho(|A_c^{-1}|\Delta) = 0.$$

Thus the interval matrix (2) is strongly regular (i.e. $\varrho(|A_c^{-1}|\Delta) < 1$); problems with strongly regular interval matrices have been usually considered "tractable".

In order to be able to formulate a unifying complexity result, we introduce the following concept: *enclosure algorithm* is an algorithm which for each system $A^I x = b^I$ with rational data (and square $A^I$) in a finite number of steps either computes a rational enclosure, or fails (i.e., issues an error message). Failure of an enclosure algorithm may be caused by various reasons: 1) no enclosure exists since the solution set is unbounded (in case of a singular $A^I$), 2) the algorithm cannot be continued (e.g. in case of the interval Gaussian algorithm), 3) the algorithm works under some condition only (e.g., strong regularity), 4) a prescribed number of steps has been reached, etc.

3. Main result

**Theorem 1.** If $P \neq NP$, then for each polynomial-time enclosure algorithm and each rational $\varepsilon > 0$, $\delta > 0$ either (i), or (ii) holds:

(i) the algorithm fails for some system in $H_{\varepsilon\delta}$,

(ii) for each rational $\alpha > 0$ and each integer $k \geq 0$ there exists a system of size $n \geq 2$ in $H_{\varepsilon\delta}$ for which the enclosure $[y_1, \overline{y}]$ computed by the algorithm satisfies

$$y_1 \leq \underline{x}_1 - \alpha n^k < \overline{x}_1 + \alpha n^k \leq \overline{y}_1.$$

**Remark 1.** 1) $P$ and NP are the well-known complexity classes. The conjecture that $P \neq NP$, although unproved, is widely believed to be true (cf. Garey and Johnson [2]). 2) If the conjecture holds true, then each polynomial-time enclosure algorithm which works for at least one family $H_{\varepsilon\delta}$ may produce arbitrarily large overestimations (4); hence, no (even arbitrarily bad) accuracy can be guaranteed to be achievable by a polynomial-time enclosure algorithm.
Proof. Assume to the contrary that there exists a polynomial-time enclosure algorithm, rational numbers \( \varepsilon > 0, \delta > 0, \alpha > 0 \) and an integer \( k \geq 0 \) such that for each system in \( H_{\varepsilon \delta} \) the algorithm computes an enclosure \([y_1, y_2]\) satisfying either

\[
\bar{x}_1 - \alpha n^k < y_1
\]

or

\[
\bar{y}_1 < \bar{x}_1 + \alpha n^k,
\]

where \( n \) is the system size. Let \( A \) be an arbitrary MC-matrix of size \( m \). Let us construct an \((m+1) \times (m+1)\) interval matrix

\[
A^I = \begin{pmatrix}
\frac{\varepsilon \delta}{\gamma} & [-\varepsilon e^T, \varepsilon e^T] \\
0 & A^{-1}
\end{pmatrix},
\]

where

\[
\gamma = \alpha (m+1)^k,
\]

and an \((m+1)\)-dimensional interval vector

\[
b^I = \begin{pmatrix}
0 \\
[-\delta e, \delta e]
\end{pmatrix},
\]

and apply the algorithm to the system

\[
A^I x = b^I
\]

(which obviously belongs to \( H_{\varepsilon \delta} \)) to compute an enclosure \([y_1, y_2]\) which, according to the assumption, satisfies either

\[
\bar{x}_1 - \gamma < y_1
\]

or

\[
\bar{y}_1 < \bar{x}_1 + \gamma.
\]

This can be done in polynomial time. We shall prove that

\[
\|A\|_{\infty,1} = \left[ \frac{1}{\gamma} \min\{-y_1, y_1\} \right]
\]

holds, where \([\ldots]\) denotes the integer part. Hence, \(\|A\|_{\infty,1}\) can be computed in polynomial time; but since this is an NP-hard problem (Proposition 1), P=NP will follow. To prove (8), first observe that the system (5) can be written as

\[
\frac{\varepsilon \delta}{\gamma} x_1 + [-\varepsilon e^T, \varepsilon e^T] x' = 0,
\]

\[-\delta e \leq A^{-1} x' \leq \delta e,
\]

where \( x' = (x_2, \ldots, x_m) \). Hence

\[
\bar{x}_1 = \frac{\gamma}{\varepsilon \delta} \max\{\varepsilon e^T | x' |; \ -\delta e \leq A^{-1} x' \leq \delta e\}
\]

\[
= \gamma \max\{\|Ax''\|_1; \ -e \leq A^{-1} x'' \leq e\}
\]

\[
= \gamma \max\{\|Ax''\|_1; \ \|x''\|_\infty = 1\}
\]

and in a quite similar way,

\[
\bar{y}_1 = -\gamma \|A\|_{\infty,1}.
\]

Hence from (6) and (7) we obtain that either

\[
-\frac{1}{\gamma} y_1 < \|A\|_{\infty,1} + 1
\]
or
\[ \frac{1}{\gamma} \gamma_1 < \|A\|_{\infty, 1} + 1 \]
holds, in both the cases
\[ \frac{1}{\gamma} \min \{ -y_1, \bar{y}_1 \} < \|A\|_{\infty, 1} + 1. \] (9)

But since \([\underline{y}_1, \bar{y}_1]\) encloses \([\underline{x}_1, \bar{x}_1]\), from \(\underline{y}_1 \leq \underline{x}_1, \bar{x}_1 \leq \bar{y}_1\) we have
\[ \|A\|_{\infty, 1} \leq \frac{1}{\gamma} \min \{ -y_1, \bar{y}_1 \} \]
which together with (9) gives
\[ \|A\|_{\infty, 1} \leq \frac{1}{\gamma} \min \{ -y_1, \bar{y}_1 \} < \|A\|_{\infty, 1} + 1. \] (10)

However, the number
\[ \|A\|_{\infty, 1} = \max \{ \|Ax\|_1; \|x\|_\infty = 1 \} = \max \{ \|Ax\|_1; x_j \in \{-1, 1\} \text{ for each } j \} \]
is integer for an MC-matrix \(A\) (which is integer by definition), hence from (10) we finally obtain that
\[ \|A\|_{\infty, 1} = \left[ \frac{1}{\gamma} \min \{ -y_1, \bar{y}_1 \} \right], \]
which is (8). Hence, \(\|A\|_{\infty, 1}\) can be computed in polynomial time for an MC-matrix \(A\), which in view of Proposition 1 implies that P=NP. This concludes the proof by contradiction.

4. Application: interval Gaussian algorithm

For each rational \(\varepsilon > 0, \delta > 0\), the interval Gaussian algorithm with partial pivoting \([1]\) (which is polynomial-time) is performable for each system in \(H_{\varepsilon \delta}\) since all the pivots are real and nonzero due to the special form of the system matrix (2). Hence, if P\(\neq\)NP, then arbitrarily large overestimations (4) may occur for arbitrarily narrow system matrices (2) and arbitrarily narrow right-hand sides (3).

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5. References


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