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Validd Solutions of Nonlinear Equations

We give an existence and uniqueness check for systems of nonlinear equations together with an iterative method which yields a validated enclosure of the solution at each iteration.

1. The result

We consider here a system of $n$ nonlinear equations in $n$ unknowns of the form

$$x = F(x)$$

(1)

over an $n$-dimensional hyperrectangle $[\hat{x} - d, \hat{x} + d] = \{x'; \hat{x} - d \leq x' \leq \hat{x} + d\}$. Existence theorems for (1) were given by Miranda, Kantorovich, Smale and others (see the recent paper by Alefeld, Gienger and Potra [1] for a survey and a list of references). The following theorem gives an existence and uniqueness check and an iterative method which yields a validated enclosure of the solution of (1) at each iteration:

**Theorem 1:** Let $F$ map a hyperrectangle $[\hat{x} - d, \hat{x} + d] \subset \mathbb{R}^n$ into $\mathbb{R}^n$, and let there exist a nonnegative matrix $H$ with the following properties:

(i) $|F(x') - F(x'')| \leq H|x' - x''|$ for each $x', x'' \in [\hat{x} - d, \hat{x} + d]$,

(ii) $|\hat{x} - F(\hat{x})| < (I - H)d$.

Then the equation (1) has a unique solution $x^*$ in $[\hat{x} - d, \hat{x} + d]$, and the sequences $\{x_j\}_{j=0}^\infty$ and $\{d_j\}_{j=0}^\infty$ given by

$$x_{j+1} = F(x_j),$$

$$d_{j+1} = Hd_j$$

(3)

for $j = 0, 1, \ldots$, $x_0 = \hat{x}$, $d_0 = d$, satisfy $x_j \to x^*$, $d_j \to 0$ and

$$x^* \in [x_j - d_j, x_j + d_j]$$

(4)

for each $j$. Moreover, the sequence

$$\{[x_j - d_j, x_j + d_j]\}_{j=0}^\infty$$

(5)

is nested.

**Proof:** 1) Since $H$ is nonnegative, from (ii) we have $Hd < d$ and $d > 0$, hence $g(H) < 1$, $(I - H)^{-1} \geq 0$, and $H^3 \to 0$ (Neumaier [2], sect. 3.2), thus also $d_j = H^3d \to 0$.

2) We shall prove by induction that

$$x_j \in [\hat{x} - d, \hat{x} + d]$$

(6)

for each $j$ (hence the sequence $\{x_j\}$ is well defined by (2)) and

$$|x_j - x_{j+1}| \leq d_j - d_{j+1}$$

(7)

for each $j$. For $j = 0$ we obviously have $x_0 = \hat{x} \in [\hat{x} - d, \hat{x} + d]$ and $|x_0 - x_1| = |\hat{x} - F(\hat{x})| < (I - H)d = d - d_1$ due to (ii). Assume that (6) and (7) hold for $j = 0, \ldots, k - 1$. Then $|\hat{x} - x_k| = |\sum_{j=0}^{k-1}(x_{j+1} - x_{j+1})| \leq \sum_{j=0}^{k-1}|x_j - x_{j+1}| \leq \sum_{j=0}^{k-1}(d_j - d_{j+1}) = d_0 - d_k \leq 0 = d$ (since $d_k \geq 0$ due to (3)), hence $x_k \in [\hat{x} - d, \hat{x} + d]$. Next, by (i) and by (7) for $j = k - 1$ we obtain $|x_k - x_{k+1}| = |F(x_{k-1}) - F(x_k)| \leq H|x_{k-1} - x_k| \leq H(d_{k-1} - d_k) = d_k - d_{k+1}$, which concludes the inductive proof of (6) and (7). Since $\{d_j\}$ is decreasing by (7) and $d_j \to 0$ (cf. 1)), we have that $d_j \to 0$. 


3) For each \( j \geq 0 \) and \( m \geq 1 \), from (7) we have \(|x_j - x_{j+m}| \leq \sum_{k=j}^{j+m-1}|x_k - x_{k+1}| \leq \sum_{k=j}^{j+m-1}(d_k - d_{k+1}) = d_j - d_{j+m} \), hence
\[
|x_j - x_{j+m}| \leq d_j - d_{j+m}.
\] (8)
Since \( \{d_j\} \) is convergent, for each positive vector \( \varepsilon > 0 \) there exists a \( j \geq 0 \) such that \(|d_j - d_{j+m}| = d_j - d_{j+m} < \varepsilon \) for each \( m \geq 1 \). Then (8) gives that \(|x_j - x_{j+m}| < \varepsilon \), hence \( \{x_j\} \) is Cauchian, so that \( x_j \to x^* \), and (6) implies that \( x^* \in [\hat{x} - d, \hat{x} + d] \).

4) Since \( F \) is continuous in \([\hat{x} - d, \hat{x} + d]\) due to (1), taking \( j \to \infty \) in (2), we obtain \( x^* = F(x^*) \). Let \( \hat{x} \) be any other solution to (1) in \([\hat{x} - d, \hat{x} + d]\). Then from \(|\hat{x} - x^*| = |F(\hat{x}) - F(x^*)| \leq H|\hat{x} - x^*| \) (due to (i)) we have \((I - H)|\hat{x} - x^*| \leq 0\), and premultiplying this inequality by the nonnegative matrix \((I - H)^{-1}\) (cf. 1) yields \(|\hat{x} - x^*| \leq 0\), hence \( \hat{x} = x^* \). Thus \( x^* \) is the unique solution of (1) in \([\hat{x} - d, \hat{x} + d]\).

5) For each \( j \geq 0 \), taking \( m \to \infty \) in (8), we obtain \(|x_j - x^*| \leq d_j\), hence \( x^* \in [x_j - d_j, x_j + d_j] \), which proves (4). From (7) it follows that \( x_j - d_j \leq x_{j+1} - d_{j+1} \) and \( x_{j+1} + d_{j+1} \leq x_j + d_j \) for each \( j \), hence the sequence of hyperrectangles (5) is nested. This completes the proof.

In practice, the original problem is usually given in the form \( f(x) = 0 \), which is brought to the form (1) by employing a mapping \( F(x) = x - Rf(x) \), where \( R \) is some nonsingular matrix. The second theorem shows that under mild assumptions the conditions (i), (ii) imposed on \( F \) in Theorem 1 are satisfied in a neighbourhood of the solution \( x^* \). The Jacobian matrices of \( f \) and \( F \) are denoted by \( J_f, J_F \), respectively, and \( \rho \) is the spectral radius:

**Theorem 2:** Let \( f(x^*) = 0 \) and let \( f \) have continuous partial derivatives in a neighbourhood of \( x^* \). Then for each \( n \times n \) matrix \( R \) satisfying
\[
\rho(|I - RJ_f(x^*)|) < 1
\] (9)
there exists a \( d > 0 \) such that the mapping
\[
F(x) = x - Rf(x)
\] (10)
satisfies the assumptions (i), (ii) of Theorem 1 in \([x^* - d, x^* + d]\).

**Proof:** First, the mapping \( F \) given by (10) obviously satisfies \( J_F(x) = I - RJ_f(x) \), hence (9) gives \( \rho(|J_F(x^*)|) < 1 \). Let \( d > 0 \) be such that \( f \) has continuous partial derivatives in \([x^* - d, x^* + d]\). For each \( d' \) satisfying \( 0 \leq d' \leq \hat{d} \) define a matrix \( H(d') = (h_{ij}(d')) \) by
\[
(h_{ij}(d')) = \max \left\{ \left| \frac{\partial F_i}{\partial x_j} \right| : x \in [x^* - d', x^* + d'] \right\}
\] (11)
for \( i, j = 1, \ldots, n \). Then \( H(0) = |J_F(x^*)| \) and \( \rho(H(0)) = \rho(|J_F(x^*)|) < 1 \), hence in view of continuity of the spectral radius there exists a \( d' > 0 \) with \( d' \leq \hat{d} \) such that \( \rho(H(d')) < 1 \). Since \( H(d') \) is nonnegative, there exists a vector \( d'' > 0 \) satisfying \( H(d')d'' < d'' \) (Neumaier [2]). Take a sufficiently small real number \( \alpha > 0 \) such that \( \alpha d'' \leq d' \), and put \( d = \alpha d'' \). Then \( d \leq d' \) implies \( H(d) \leq H \) by (11), and from \( H(d')d'' < d'' \) we have \( Hd < d' \). Hence for each \( x', x'' \in [x^* - d, x^* + d] \) we have by the mean-value theorem and by (11) that \(|F(x') - F(x'')| \leq H(d)|x' - x''| \leq H|x' - x''| \) and \(|x^* - F(x^*)| = 0 < (I - H)d \) hold, hence the assumptions (i) and (ii) of Theorem 1 are satisfied in \([x^* - d, x^* + d]\).

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**References**


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