Computing the Norm $\|A\|_{\infty,1}$ is NP-Hard

Dedicated to Professor Svatopluk Poljak, in memoriam

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Abstract
It is proved that computing the subordinate matrix norm $\|A\|_{\infty,1}$ is NP-hard. Even more, existence of a polynomial-time algorithm for computing this norm with relative accuracy less than $1/(4n^2)$, where $n$ is matrix size, implies $P=NP$.

Key words. Norm, positive definiteness, $M$-matrix, NP-hardness

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1 Introduction
Given two vector norms $\|x\|_\alpha$ in $\mathbb{R}^n$ and $\|x\|_\beta$ in $\mathbb{R}^m$, a subordinate matrix norm in $\mathbb{R}^{m\times n}$ is defined by

$$\|A\|_{\alpha,\beta} = \max_{\|x\|_\alpha=1} \|Ax\|_\beta$$

(Golub and van Loan [4]). $\|A\|_{\alpha,\beta}$ is a matrix norm, i.e., it possesses the three usual properties: 1) $\|A\|_{\alpha,\beta} \geq 0$ and $\|A\|_{\alpha,\beta} = 0$ if and only if $A = 0$, 2) $\|A + B\|_{\alpha,\beta} \leq \|A\|_{\alpha,\beta} + \|B\|_{\alpha,\beta}$, 3) $\|\lambda A\|_{\alpha,\beta} = |\lambda| \cdot \|A\|_{\alpha,\beta}$. However, generally it does not possess the property $\|AB\|_{\alpha,\beta} \leq \|A\|_{\alpha,\beta} \cdot \|B\|_{\alpha,\beta}$ (it does e.g. if $\alpha = \beta$).

By combining the two frequently used norms

$$\|x\|_1 = \sum_i |x_i|,$$
$$\|x\|_\infty = \max_i |x_i|,$$

we get three well-known easily computable subordinate norms

$$\|A\|_{1,1} = \max_j \sum_i |a_{ij}|,$$
$$\|A\|_{\infty,\infty} = \max_i \sum_j |a_{ij}|,$$
$$\|A\|_{1,\infty} = \max_{ij} |a_{ij}|$$

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(see Golub and van Loan [4]). It turns out, however, that the fourth norm

\[ \|A\|_{\infty,1} = \max_{\|x\|_{\infty}=1} \|Ax\|_1 \]

has an exceptional behavior since it is much more difficult to compute. In this paper we prove that computing \(\|A\|_{\infty,1}\) is NP-hard. For the purpose of various applications, the result is presented in several different settings (Theorems 3 to 6).

2 The norm \(\|A\|_{\infty,1}\)

This norm can be computed by a finite formula which, however, involves maximization over the set \(Z\) of all \(\pm 1\)-vectors of length \(n\) (whose cardinality is \(2^n\)):

**Proposition 1** For each \(A \in \mathbb{R}^{m \times n}\) we have

\[ \|A\|_{\infty,1} = \max_{z \in Z} \|Az\|_1, \tag{1} \]

where \(Z = \{z \in \mathbb{R}^n; z_j \in \{-1, 1\} \text{ for each } j\}\).

Moreover, if \(A\) is symmetric positive semidefinite, then

\[ \|A\|_{\infty,1} = \max_{z \in Z} z^T Az. \tag{2} \]

**Proof.** 1) If \(\|x\|_{\infty} = 1\), then \(x\) belongs to the unit cube \(\{x; -e \leq x \leq e\}\), \(e = (1, \ldots, 1)^T\), which is a convex polyhedron, therefore \(x\) can be expressed as a convex combination of its vertices which are exactly the points in \(Z\):

\[ x = \sum_{z \in Z} \lambda_z z, \tag{3} \]

where \(\lambda_z \geq 0\) for each \(z \in Z\) and \(\sum_{z \in Z} \lambda_z = 1\). From (3) we have

\[ \|Ax\|_1 = \| \sum_{z \in Z} \lambda_z Az \|_1 \leq \max_{z \in Z} \|Az\|_1, \]

hence

\[ \max_{\|x\|_{\infty}=1} \|Ax\|_1 \leq \max_{z \in Z} \|Az\|_1 \leq \max_{\|x\|_{\infty}=1} \|Ax\|_1 \]

(since \(\|z\|_{\infty} = 1\) for each \(z \in Z\)), and (1) follows.

2) Let \(A\) be symmetric positive semidefinite and let \(z \in Z\). Define \(y \in Z\) by \(y_j = 1\) if \((Az)_j \geq 0\) and \(y_j = -1\) if \((Az)_j < 0\) \((j = 1, \ldots, n)\), then

\[ \|Az\|_1 = y^T Az. \]
Since $A$ is symmetric positive semidefinite, we have
\[(y - z)^T A (y - z) \geq 0,\]
which implies
\[2y^T Az \leq y^T Ay + z^T Az \leq 2 \max_{z \in Z} z^T Az,\]
hence
\[\|Az\|_1 = y^T Az \leq \max_{z \in Z} z^T Az\]
and
\[\|A\|_{\infty, 1} = \max_{z \in Z} \|Az\|_1 \leq \max_{z \in Z} z^T Az.\]  (4)

Conversely, for each $z \in Z$ we have
\[z^T Az \leq |z^T Az| \leq |z|^T |Az| = e^T |Az| = \|Az\|_1 \leq \max_{z \in Z} \|Az\|_1 = \|A\|_{\infty, 1},\]
hence
\[\max_{z \in Z} z^T Az \leq \|A\|_{\infty, 1},\]
which together with (4) gives (2).

A weaker form of (2) ($\|A\|_{\infty, 1} = \max_{\|x\|_\infty = 1} x^T Ax$) was given by Tao [8]. In section 4 we shall prove that computing $\|A\|_{\infty, 1}$ is NP-hard. This suggests that unless P=NP, the formulae (1), (2) cannot be essentially simplified.

3 MC-matrices

In order to prove the NP-hardness for a suitably narrow class of matrices, we introduce the following concept (first formulated in [7]):

\textbf{Definition 1} A symmetric $n \times n$ matrix $A = (a_{ij})$ is called an MC-matrix\(^1\) if it is of the form
\[a_{ij} \begin{cases} = n & \text{if } i = j, \\ \in \{0, -1\} & \text{if } i \neq j \end{cases}\]
(i, j = 1, \ldots, n).

Since an MC-matrix is symmetric by definition, there are altogether $2^{n(n-1)/2}$ MC-matrices of size $n$. The basic properties of MC-matrices are summed up in the following proposition (where we denote, as customary, by $\|A\|_1$ the norm $\|A\|_{1,1}$ described in Section 1):

\(^1\)from “maximum cut”; see the proof of Theorem 3 below

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Proposition 2 An MC-matrix $A \in \mathbb{R}^{n \times n}$ is symmetric positive definite, nonnegative invertible and satisfies

$$\|A\|_{\infty,1} = \max_{z \in \mathbb{Z}} z^T A z,$$ 

(5)

$$n \leq \|A\|_{\infty,1} \leq n(2n - 1)$$

(6)

and

$$\|A^{-1}\|_1 \leq 1.$$ 

Proof. $A$ is symmetric by definition; it is positive definite since for $x \neq 0$,

$$x^T A x \geq n \|x\|_2^2 - \sum_{i \neq j} |x_i x_j| = (n + 1) \|x\|_2^2 - \|x\|_1^2 \geq \|x\|_2^2 > 0$$

($\|x\|_1 \leq \sqrt{n} \|x\|_2$ by the Cauchy-Schwarz inequality [4]). Hence (5) holds by Proposition 1. Since $|a_{ij}| \leq 1$ for $i \neq j$, for each $z \in \mathbb{Z}$ and each $i \in \{1, \ldots, n\}$ we have

$$z_i (Az)_i = n + \sum_{j \neq i} a_{ij} z_i z_j \in [1, 2n - 1],$$

hence

$$n \leq z^T A z \leq n(2n - 1)$$

for each $z \in \mathbb{Z}$, and (5) implies (6). Putting

$$A_0 = nI - A,$$

we have that $A_0 \geq 0$, $A = nI - A_0 = n(I - \frac{1}{n} A_0)$ and $\|\frac{1}{n} A_0\|_1 \leq \frac{n-1}{n} < 1$, hence

$$A^{-1} = \frac{1}{n} \sum_{0}^{\infty} (\frac{1}{n} A_0)^j \geq 0$$

and

$$\|A^{-1}\|_1 \leq \frac{1}{n - \|A_0\|_1} \leq 1,$$

which completes the proof. 

Hence an MC-matrix $A \in \mathbb{R}^{n \times n}$ satisfies

$$\|A\|_1 \cdot \|A^{-1}\|_1 < 2n,$$

i.e., it is well conditioned.
4 Computing $\|A\|_{\infty,1}$ is NP-hard

The following basic result is due to Poljak and Rohn [5] (given there in another formulation without using the concept of an $MC$-matrix).

**Theorem 3** The following decision problem is NP-complete:

Instance. An $MC$-matrix $A$ and a positive integer $\ell$.

Question. Is $z^T A z \geq \ell$ for some $z \in \mathbb{Z}$?

**Proof.** Let $(N, E)$ be an undirected graph with $N = \{1, \ldots, n\}$. Let $A = (a_{ij})$ be given by $a_{ij} = n$ if $i = j$, $a_{ij} = -1$ if $i \neq j$ and the nodes $i, j$ are connected by an edge, and $a_{ij} = 0$ if $i \neq j$ and $i, j$ are not connected. Then $A$ is an $MC$-matrix. For $S \subseteq N$, define the cut $c(S)$ as the number of edges in $E$ whose one endpoint belongs to $S$ and the other one to $N - S$. We shall prove that

$$\|A\|_{\infty,1} = 4 \max_{S \subseteq N} c(S) - 2|E| + n^2$$

holds, where $|E|$ denotes the cardinality of $E$. Given a $S \subseteq N$, define a $z \in \mathbb{Z}$ by

$$z_i = \begin{cases} 1 & \text{if } i \in S, \\ -1 & \text{if } i \notin S. \end{cases}$$

Then we have

$$z^T A z = \sum_{i,j} a_{ij} z_i z_j = \sum_{i \neq j} a_{ij} z_i z_j + n^2$$

$$= \sum_{i \neq j} [-\frac{1}{2} a_{ij} (z_i - z_j)^2 + a_{ij}] + n^2$$

$$= -\frac{1}{2} \sum_{z_i, z_j = -1} a_{ij} (z_i - z_j)^2 + \sum_{i \neq j} a_{ij} + n^2$$

$$= -\frac{1}{2} \sum_{z_i, z_j = -1} 4a_{ij} + \sum_{i \neq j} a_{ij} + n^2,$$

hence

$$z^T A z = 4c(S) - 2|E| + n^2.$$ (8)

Conversely, given $z \in \mathbb{Z}$, then for $S = \{i \in N; z_i = 1\}$ the same reasoning implies (8). Taking maximum on both sides of (8), we obtain (7) in view of (5).

Hence, given a positive integer $L$, we have that

$$c(S) \geq L$$

is valid for some $S \subseteq N$ if and only if

$$z^T A z \geq 4L - 2|E| + n^2$$
holds for some \( z \in \mathbb{Z} \). Since the decision problem (9) is NP-complete ("simple max-cut problem", Garey, Johnson and Stockmeyer [3]), we obtain that the decision problem

\[
\begin{align*}
    z^T A z & \geq \ell \\
\end{align*}
\]

(\( \ell \) positive integer) is NP-hard. It is NP-complete since for a guessed solution \( z \in \mathbb{Z} \) the validity of (10) can be checked in polynomial time.

In this way, in view of (5) we have also proved the following result:

**Theorem 4** Computing \( \| A \|_{\infty,1} \) is NP-hard in the class of MC-matrices.

To facilitate formulations of some applications of these results [6], it is advantageous to remove the integer parameter \( \ell \) from the formulation of Theorem 3. This can be done by using \( M \)-matrices instead of \( MC \)-matrices. Let us recall that \( A = (a_{ij}) \) is called an \( M \)-matrix if \( a_{ij} \leq 0 \) for \( i \neq j \) and \( A^{-1} \geq 0 \) (a number of equivalent formulations can be found in Berman and Plemmons [1]); hence each \( MC \)-matrix is an \( M \)-matrix (Proposition 2). Since a symmetric \( M \)-matrix is positive definite [1], this property is not explicitly mentioned in the following theorem:

**Theorem 5** The following decision problem is NP-hard:

**Instance.** A symmetric rational \( M \)-matrix \( A \).

**Question.** Is \( \| A \|_{\infty,1} \geq 1 \)?

**Proof.** Given an \( MC \)-matrix \( A \) and a positive integer \( \ell \), the assertion

\[
    z^T A z \geq \ell \text{ for some } z \in \mathbb{Z}
\]

is equivalent to \( \| A \|_{\infty,1} \geq \ell \) and thereby also to

\[
    \left\| \frac{1}{\ell} A \right\|_{\infty,1} \geq 1,
\]

where \( \frac{1}{\ell} A \) is a symmetric rational \( M \)-matrix. Hence the decision problem of Theorem 3 can be reduced in polynomial time to the current one, which is then NP-hard.

Finally we shall show that even computing a sufficiently close approximation of \( \| A \|_{\infty,1} \) is NP-hard:

**Theorem 6** Suppose there exists a polynomial-time algorithm which for each \( MC \)-matrix \( A \) computes a rational number \( \nu(A) \) satisfying

\[
    \left| \frac{\nu(A) - \| A \|_{\infty,1}}{\| A \|_{\infty,1}} \right| \leq \frac{1}{4n^2},
\]

where \( n \) is the size of \( A \). Then \( P=NP \).
Proof. If such an algorithm exists, then
\[ |\nu(A) - \|A\|_{\infty,1}| \leq \frac{\|A\|_{\infty,1}}{4n^2} \leq \frac{n(2n-1)}{4n^2} < \frac{1}{2} \]
due to (6), which implies
\[ \|A\|_{\infty,1} < \nu(A) + \frac{1}{2} < \|A\|_{\infty,1} + 1, \]
and thereby also
\[ \|A\|_{\infty,1} = \left\lfloor \nu(A) + \frac{1}{2} \right\rfloor \]
(since \( \|A\|_{\infty,1} \) is integer for an MC-matrix \( A \) by (5)). Hence the NP-hard problem of Theorem 4 can be solved in polynomial time, implying \( P=NP \).

Various applications of Theorem 5 for problems with inexact data (regularity, positive definiteness, stability, solvability of linear equations and inequalities, linear and quadratic programming) are given in [6].

5 Concluding remarks

We have proved that existence of a polynomial-time algorithm for computing \( \|A\|_{\infty,1} \) with relative accuracy less than \( \frac{1}{4n^2} \) implies that the complexity classes \( P \) and \( NP \) are equal. This runs against the famous unproved conjecture that \( P \neq NP \) holds, which is widely believed to be true (see Garey and Johnson [2] for details). Hence, the existence of such a polynomial-time algorithm seems highly unlikely, although it cannot be ruled out by current results in complexity theory.

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References


