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Abstract:

We describe a not-a-priori-exponential necessary and sufficient condition for regularity of interval matrices which is an easy consequence of an earlier result on interval linear equations.

Keywords:
Interval matrix, regularity, necessary and sufficient condition.

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⁴Above: logo of interval computations and related areas (depiction of the solution set of the system \([2, 4]x_1 + [-2, 1]x_2 = [-2, 2], [-1, 2]x_1 + [2, 4]x_2 = [-2, 2]\) (Barth and Nuding [11]).
Checking regularity of interval matrices is a known NP-hard problem. Forty necessary and sufficient regularity conditions are summed up in [2]; all of them are exponential because they explicitly or implicitly contain the quantifier “for each $z \in \{-1, 1\}^n$”. The condition given below is, to these authors’ knowledge, the first ever published not-a-priori-exponential regularity condition because instead of $\{-1, 1\}^n$ it employs only a subset $Z$ of it. Cardinality of the set $Z$ varies with the data, but its minimal value is 1. Notation used: $e_j$ is the $j$th column of the $n \times n$ identity matrix, $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$, diag($z$) is the $n \times n$ diagonal matrix with diagonal vector $z$ and for an $x \in \mathbb{R}^n$, sgn($x$) is defined by (sgn($x$))$_i = 1$ if $x_i \geq 0$ and (sgn($x$))$_i = -1$ otherwise.

**Theorem 1.** An $n \times n$ interval matrix $A = [A_c - \Delta, A_c + \Delta]$ is regular if and only if $A_c$ is nonsingular and there exists a subset $Z$ of $\{-1, 1\}^n$ having the following properties:

(a) $\text{sgn}(A_c^{-1}e) \in Z$,

(b) for each $z \in Z$ the inequalities

$$
(QA_c - I) \text{ diag}(z) \geq |Q|\Delta, \quad \text{(0.1)}
$$

$$
(QA_c - I) \text{ diag}(-z) \geq |Q|\Delta \quad \text{(0.2)}
$$

have matrix solutions $Q_z$ and $Q_{-z}$, respectively,

(c) if $z \in Z$, $Q_z e \leq Q_{-z} e$, and $(Q_{-z} e)_j(Q_z e)_j \leq 0$ for some $j$, then $z - 2z_j e_j \in Z$.

**Proof.** “If”: The assumptions (a)-(c) imply that the three assumptions of Theorem 3 in [3] are met for the system of interval linear equations $Ax = [e, e]$ whose solution set in virtue of the same theorem is bounded, hence $A$ is regular. “Only if”: If $A$ is regular, then (a) and (c) are satisfied for $Z = \{-1, 1\}^n$ and for each $z \in \{-1, 1\}^n$ the equations

$$
(QA_c - I) \text{ diag}(z) = |Q|\Delta,
$$

$$
(QA_c - I) \text{ diag}(-z) = |Q|\Delta
$$

have (even unique) solutions, see [2].

Hence we can also formulate the theorem in the following way:

**Theorem 2.** An $n \times n$ interval matrix $A = [A_c - \Delta, A_c + \Delta]$ is regular if and only if $A_c$ is nonsingular and there exists a subset $Z$ of $\{-1, 1\}^n$ having the following properties:

(a) $\text{sgn}(A_c^{-1}e) \in Z$,

(b) for each $z \in Z$ the equations

$$
(QA_c - I) \text{ diag}(z) = |Q|\Delta, \quad \text{(0.3)}
$$

$$
(QA_c - I) \text{ diag}(-z) = |Q|\Delta \quad \text{(0.4)}
$$

have matrix solutions $Q_z$ and $Q_{-z}$, respectively,

(c) if $z \in Z$, $Q_z e \leq Q_{-z} e$, and $(Q_{-z} e)_j(Q_z e)_j \leq 0$ for some $j$, then $z - 2z_j e_j \in Z$.

Notice that if $z \in \{-1, 1\}^n$, then $z - 2z_j e_j \in \{-1, 1\}^n$ (in (c)), so that $Z \subseteq \{-1, 1\}^n$; thus $Z$ is defined recursively by (a) and (c). In practical computations, equations (0.3), (0.4) are solved instead of inequalities (0.1), (0.2) as it was done in the function `qzmatrix`, using the subfunction `absvaleqn`, in [4].
Bibliography


