Overdetermined Absolute Value Equations

Jiří Rohn
http://uivtx.cs.cas.cz/~rohn

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Abstract:
We consider existence, uniqueness and computation of a solution of an absolute value equation in the overdetermined case.\textsuperscript{2}

Keywords:
Absolute value equations, overdetermined system.

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\textsuperscript{2}Above: logo of interval computations and related areas (depiction of the solution set of the system $[2,4]x_1 + [-2,1]x_2 = [-2,2]$, $[-1,2]x_1 + [2,4]x_2 = [-2,2]$ (Barth and Nuding [1])).
1 Introduction

The absolute value equation
\[ Ax + B|x| = b \]  
(1.1)
has been studied so far for the square case only \((A, B \in \mathbb{R}^{n \times n})\). In this report we consider the rectangular case \((A, B \in \mathbb{R}^{m \times n})\); the assumption (2.1) made below ensures that \(m \geq n\), so that in fact we investigate the overdetermined case only.

Notation used: \(|x|\) is the entrywise absolute value of \(x\), \(\rho\) denotes the spectral radius, \(I\) is the identity matrix and \(A^\dagger\) stands for the Moore-Penrose inverse of \(A\).

2 The result

We shall handle the questions of existence, uniqueness and computation of a solution in frame of a single theorem.

**Theorem 1.** Let \(A, B \in \mathbb{R}^{m \times n}\) satisfy
\[ \text{rank}(A) = n \]  
(2.1)
and
\[ \rho(|A^\dagger B|) < 1. \]  
(2.2)
Then for each \(b \in \mathbb{R}^m\) the sequence \(\{x^i\}_{i=0}^\infty\) generated by
\[ x^0 = A^\dagger b, \]
\[ x^{i+1} = -A^\dagger B|x^i| + A^\dagger b \quad (i = 0, 1, 2, \ldots) \]  
(2.3)  
(2.4)
tends to a limit \(x^*\), and we have:

(i) if \(Ax^* + B|x^*| = b\), then \(x^*\) is the unique solution of (1.1),

(ii) if \(Ax^* + B|x^*| \neq b\), then (1.1) possesses no solution.

**Proof.** For clarity, we divide the proof into several steps.

(a) From (2.4) we have
\[ |x^{i+1} - x^i| \leq |A^\dagger B||x^i - x^{i-1}| \]
for each \(i \geq 1\) and since \(|A^\dagger B|^j \to 0\) as \(j \to \infty\) due to (2.2), proceeding as in the proof of Theorem 1 in [2] we prove that \(\{x^i\}\) is a Cauchian sequence, thus it is convergent, \(x^i \to x^*\).

Taking the limit in (2.4) we obtain that \(x^* = -A^\dagger B|x^*| + A^\dagger b\), i.e., \(x^*\) solves the equation
\[ x + A^\dagger B|x| = A^\dagger b. \]  
(2.5)

(b) Assume that \(\tilde{x}\) also solves (2.5). Then
\[ |x^* - \tilde{x}| \leq |A^\dagger B||x^* - \tilde{x}|, \]
hence
\[ (I - |A^\dagger B|)|x^* - \tilde{x}| \leq 0 \]
and premultiplying this inequality by the inverse of $I - A^\dagger B$ which is nonnegative due to (2.2) results in

$$|x^* - \bar{x}| \leq 0,$$

hence $x^* = \bar{x}$ which means that $x^*$ is the unique solution to (2.5).

(c) We prove that if $x$ solves (1.1), then $x = x^*$. Indeed, in that case it also solves the preconditioned equation

$$A^\dagger Ax + A^\dagger B|x| = A^\dagger b$$

and since $A^\dagger = (A^TA)^{-1}A^T$ due to (2.1), $A^\dagger A = I$ and $x$ solves (2.5) so that $x = x^*$.

(d) If $Ax^* + B|x^*| = b$, then $x^*$ is a solution of (1.1) and it is unique by (c).

(e) If $Ax^* + B|x^*| \neq b$, then existence of a solution $x$ to (1.1) would mean that $x = x^*$ by (c), hence $Ax^* + B|x^*| = b$, a contradiction.

We have this immediate consequence.

**Theorem 2.** Under conditions (2.1) and (2.2) the equation (1.1) possesses for each $b \in \mathbb{R}^m$ at most one solution.
Bibliography
