Absolute Value Mapping

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Technical report No. V-1266

05.05.2019
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Abstract:

We prove a necessary and sufficient condition for an absolute value mapping to be bijective. This result simultaneously gives a characterization of unique solvability of an absolute value equation for each right-hand side.²

Keywords:
Absolute value mapping, bijectivity, interval matrix, regularity, absolute value equation, unique solvability.

¹This work was supported with institutional support RVO:67985807.
²Above: logo of interval computations and related areas (depiction of the solution set of the system \([2, 4]x_1 + [-2, 1]x_2 = [-2, 2], [-1, 2]x_1 + [2, 4]x_2 = [-2, 2]\) (Barth and Nuding [1])).
1 Introduction

The mapping

\[ f_{AB}(x) = Ax + B|x|, \]  

(1.1)

where \( A, B \in \mathbb{R}^{n \times n} \), is called an absolute value mapping (the absolute value of a vector is understood entrywise). In this report we are solely interested in condition under which \( f_{AB} \) is bijective, i.e., is a one-to-one mapping of \( \mathbb{R}^n \) onto itself. We show below that the problem is closely connected with regularity of interval matrices.

Our result given in Theorem 3 can be also seen as a necessary and sufficient condition for unique solvability of an absolute value equation

\[ Ax + B|x| = b \]

for each right-hand side \( b \in \mathbb{R}^n \), a property for which only sufficient conditions have been known so far.

2 Auxiliary results

For the proof of the main theorem we shall need two auxiliary results that are of independent interest. Let us recall that a square matrix is called a \( P \)-matrix if all its principal minors are positive. The first result is due to Murty [2, Thm. 4.2]; \( x^+ \) and \( x^- \) are defined by \( x^+ = \max(x, 0), x^- = \max(-x, 0) \) (entrywise).

**Theorem 1.** Let \( C \in \mathbb{R}^{n \times n} \). Then the mapping

\[ g_C(x) = x^+ - Cx^- \]

is a bijection of \( \mathbb{R}^n \) onto itself if and only if \( C \) is a \( P \)-matrix.

The second result is due to Rump [4, Thm. 4.1]. The set of the form

\[ [F - G, F + G] = \{ H \mid F - G \leq H \leq F + G \} \]

where \( F, G \in \mathbb{R}^{n \times n}, G \geq 0 \), is called an interval matrix and it is said to be regular if each matrix \( H \) contained therein is nonsingular.

**Theorem 2.** Let \( C - I \) be nonsingular. Then \( C \) is a \( P \)-matrix if and only if the interval matrix

\[ [(C - I)^{-1}(C + I) - I, (C - I)^{-1}(C + I) + I] \]

is regular.

In the original Rump’s formulation nonsingularity of both \( C - I \) and \( C + I \) was assumed; it was shown later in [3, Thm 2] that the second assumption is superfluous.
3 Characterization

Assume that $A + B$ is nonsingular; then we can define the matrix

$$C = (A + B)^{-1}(A - B)$$

which satisfies

$$C - I = (A + B)^{-1}(A - B) - (A + B)^{-1}(A + B) = -2(A + B)^{-1}B,$$

$$C + I = (A + B)^{-1}(A - B) + (A + B)^{-1}(A + B) = 2(A + B)^{-1}A,$$

and $C - I$ becomes nonsingular under an additional assumption of nonsingularity of $B$. Then we can introduce a matrix $D$ by

$$D = (C - I)^{-1}(C + I) = -B^{-1}(A + B)(A + B)^{-1}A = -B^{-1}A.$$

**Theorem 3.** Let both $B$ and $A + B$ be nonsingular. Then the mapping (1.1) is a bijection of $\mathbb{R}^n$ onto itself if and only if the interval matrix

$$[D - I, D + I]$$

is regular.

**Proof.** Because $x$ and $|x|$ can be decomposed as $x = x^+ - x^-$ and $|x| = x^+ + x^-$, we have

$$f_{AB}(x) = A(x^+ - x^-) + B(x^+ + x^-) = (A + B)x^+ - (A - B)x^-$$

$$= (A + B)(x^+ - Cx^-) = (A + B)g_C(x)$$

and since $A + B$ is nonsingular, $f_{AB}$ is a bijection of $\mathbb{R}^n$ onto itself if and only if $g_C$ possesses the same property which by Theorem 1 is the case if and only if $C$ is a $P$-matrix. Now, by Theorem 2, $C$ is a $P$-matrix if and only if the interval matrix

$$[D - I, D + I]$$

is regular which concludes the proof. \qed

4 Checking

Thus checking bijectivity of $f_{AB}$ may be performed by the following MATLAB file whose subroutine can be downloaded from http://uivtx.cs.cas.cz/~rohn/other/regising.m.

```matlab
function b=bijectivity(A,B)
% b== 1: the mapping x --> A*x + B*abs(x) is bijective,
% b==-1: the mapping is not bijective.
% n=size(A,1); I=eye(n,n);
if rank(B)<n || rank(A+B)<n
    error('Condition not satisfied.')
end
D=-inv(B)*A;
S=regising(D,I);
if isempty(S), b=1; else b=-1; end
```
Bibliography


