Generalization of a Theorem on Eigenvalues of Symmetric Matrices

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Technical report No. V-1271

22.07.2019
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Abstract:
We prove that the product of a symmetric positive semidefinite matrix and a symmetric matrix has all eigenvalues real. 

Keywords:
Symmetric matrix, positive semidefinite matrix, real spectrum.

\footnote{1}{This work was supported with institutional support RVO:67985807.}
\footnote{2}{Above: Logo of interval computations and related areas (depiction of the solution set of the system \([2, 4]x_1 + [-2, 1]x_2 = [-2, 2], [-1, 2]x_1 + [2, 4]x_2 = [-2, 2]\) (Barth and Nuding [1])).}
A real matrix has complex eigenvalues in general. Yet there is a well-known important exception:

**Theorem 1.** A symmetric matrix $A \in \mathbb{R}^{n \times n}$ has all eigenvalues real.

In this note we prove a generalization of this result and we show a way how to construct generally nonsymmetric matrices having real eigenvalues only.

**Theorem 2.** If $A, B \in \mathbb{R}^{n \times n}$ are symmetric matrices of which at least one is positive semidefinite, then $AB$ has all eigenvalues real.

**Proof.** (a) First, let $A$ be positive semidefinite. Put $C = A^{1/2}$, so that $C$ is the unique symmetric positive semidefinite real matrix satisfying $C^2 = A$ (Horn and Johnson [2, Thm. 7.2.6]). Then

$$AB = C^2B = CC^TB = C(C^TB)$$

and since $C(C^TB)$ and $(C^TB)C$ have the same spectrum (Horn and Johnson [2, Thm. 1.3.20]), the eigenvalues of $AB$ and $(C^TB)C$ are equal, hence it suffices to prove that $C^TBC$ has all eigenvalues real. Thus let $\lambda \in \mathbb{C}$ be an eigenvalue of $C^TBC$, i.e.,

$$C^TBCx = \lambda x$$

holds for some $0 \neq x \in \mathbb{C}^n$ which can be normalized so that $x^*x = 1$, where $x^*$ denotes the conjugate transpose. Premultiplying (0.1) by $x^*$ yields

$$\lambda = x^*C^TBCx = y^*By = \sum_{ij} B_{ij}y_i^*y_j,$$

where $y = Cx \in \mathbb{C}^n$, and consequently

$$\lambda = \sum_i B_{ii}|y_i|^2 + \sum_{i<j} (B_{ij}y_i^*y_j + B_{ji}y_j^*y_i) = \sum_i B_{ii}|y_i|^2 + \sum_{i<j} B_{ij}(y_i^*y_j + y_j^*y_i).$$

(0.2)

Because

$$(y_i^*y_j + y_j^*y_i)^* = y_i^*y_j + y_j^*y_i,$$

the number $y_i^*y_j + y_j^*y_i$, being equal to its complex conjugate, is real and this in the light of (0.2) means that $\lambda$ is also real.

(b) If $B$ is positive semidefinite then by the part (a) above $BA$ has all eigenvalues real and thus the same holds for $AB$ whose spectrum equals to that of $BA$. \qed

To show that Theorem 2 is indeed a generalization of Theorem 1, let us decompose a given symmetric matrix $A$ as $A = AI$ where $I$ is the identity matrix, then the assumption of Theorem 2 are met which implies that all eigenvalues of $A$ are real.

Finally we describe a way how to generate generally nonsymmetric matrices having real eigenvalues only.

**Theorem 3.** For any $A, B \in \mathbb{R}^{n \times n}$, the matrices

$$A^TAB^T$$

and

$$A^T(A+B^T)$$

have all eigenvalues real.
Proof. This follows immediately from Theorem 2 since $A^T A$ is symmetric positive semidefinite, and $B^T B$ and $B + B^T$ are symmetric. \qed
Bibliography
